EM Waves & Waveguides

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The goal is to find the solutions of Maxwell's equations in the presence of some configuration of conductors. The setup of the problem is assumed to have symmetry under translation in the z-direction. Solutions are expressed via a dispersion relation $k(\omega)$ for traveling waves and a formula for the associated fields.

Maxwell's Equations

Begin with the sourceless Maxwell equations in vacuum (for linear dielectric media replace $c = 1/\sqrt{\mu_0\epsilon_0}$ with $c' = c/n = 1/\sqrt{\mu\epsilon}$). These imply the wave equation for all components of the fields.

$$\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{E}} = 0 \qquad \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{B}} = 0 \tag{1}$$

$$\boldsymbol{\nabla}\times\boldsymbol{\mathcal{E}} = -\frac{\partial\boldsymbol{\mathcal{B}}}{\partial t} \qquad \boldsymbol{\nabla}\times\boldsymbol{\mathcal{B}} = -\frac{1}{c^2}\frac{\partial\boldsymbol{\mathcal{E}}}{\partial t} \tag{2}$$

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\begin{Bmatrix}\boldsymbol{\mathcal{E}}\\ \boldsymbol{\mathcal{B}}\end{Bmatrix} = 0 \tag{3}$$

Assume the fields go as $\boldsymbol{\mathcal{A}} = \mathbf{A}e^{ikz}e^{-i\omega t}$ and break vectors up into longitudinal and transverse components, where the coefficients $\mathbf{A} = \mathbf{A_t} + \hat{\mathbf{z}}A_z$ are complex, so that

$$\boldsymbol{\mathcal{E}}(\mathbf{x},t) = [\mathbf{E_t}(x,y) + \hat{\mathbf{z}}E_z(x,y)]e^{ikz}e^{-i\omega t} \tag{4}$$

$$\boldsymbol{\mathcal{B}}(\mathbf{x},t) = [\mathbf{B_t}(x,y) + \hat{\mathbf{z}}B_z(x,y)]e^{ikz}e^{-i\omega t} \tag{5}$$

$$\boldsymbol{\nabla} = \boldsymbol{\nabla_t} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \tag{6}$$

Maxwell's equations and the wave equation become

$$\boldsymbol{\nabla_t}\cdot\mathbf{E_t} + ikE_z = 0 \tag{7}$$

$$\boldsymbol{\nabla_t}\cdot\mathbf{B_t} + ikB_z = 0 \tag{8}$$

$$\boldsymbol{\nabla_t}\times\mathbf{E_t} = i\omega B_z\hat{\mathbf{z}} \tag{9}$$

$$\boldsymbol{\nabla_t}\times\mathbf{B_t} = -i\tfrac{\omega}{c^2}E_z\hat{\mathbf{z}} \tag{10}$$

$$ik\mathbf{E_t} + i\omega(\hat{\mathbf{z}}\times\mathbf{B_t}) = \boldsymbol{\nabla_t}E_z \tag{11}$$

$$ik\mathbf{B_t} - i\tfrac{\omega}{c^2}(\hat{\mathbf{z}}\times\mathbf{E_t}) = \boldsymbol{\nabla_t}B_z \tag{12}$$

$$[\nabla_t^2 + (\frac{\omega^2}{c^2} - k^2)]\begin{Bmatrix}\mathbf{E}\\ \mathbf{B}\end{Bmatrix} = (\nabla_t^2 + \gamma^2)\begin{Bmatrix}\mathbf{E}\\ \mathbf{B}\end{Bmatrix} = 0 \tag{13}$$

The equations for the fields have reduced to equations for the coefficients. Eqns (7,8) are the divergence equations. Eqns (9,11) [(10,12)] come from the curl of E [B] equation. Eqns (9,10) are exactly the z-component of the curl equations. Meanwhile (11,12) are obtained from $\{\hat{\mathbf{z}}\times[\text{Eqn 2}]\}$. Note each term of $\{(\boldsymbol{\nabla_t} + \hat{\mathbf{z}}\frac{\partial}{\partial z})\times(\mathbf{E_t} + \hat{\mathbf{z}}E_z)\}$ is either $(\perp z)$, $(\parallel z)$, or zero.

Equations (11,12) can be decoupled, so long as $(\frac{\omega^2}{c^2} - k^2) \neq 0$, by the operations $\{[\text{Eqn 11}] - (\frac{\omega}{k}\hat{\mathbf{z}}\times[\text{Eqn 12}])\}$ and $\{[\text{Eqn 12}] + (\frac{\omega}{kc^2}\hat{\mathbf{z}}\times[\text{Eqn 11}])\}$, yielding

$$\mathbf{E_t} = \frac{i}{(\frac{\omega^2}{c^2} - k^2)}[k\boldsymbol{\nabla_t} E_z - \omega(\hat{\mathbf{z}} \times \boldsymbol{\nabla_t} B_z)] \tag{14}$$

$$\mathbf{B_t} = \frac{i}{(\frac{\omega^2}{c^2} - k^2)}[k\boldsymbol{\nabla_t} B_z - \frac{\omega}{c^2}(\hat{\mathbf{z}} \times \boldsymbol{\nabla_t} E_z)] \tag{15}$$

If E_z,B_z aren't both zero, and $(\frac{\omega^2}{c^2} - k^2) \neq 0$, then these equations (14,15) show that the fields are completely determined by specifying E_z and B_z. The case $E_z = B_z = 0$ implies $\omega = ck$ leaving (14,15) indeterminate.

These forms (7-15) of Maxwell's equations, when combined with appropriate boundary conditions, determine the mode structure of waveguides.

Boundary Conditions at Perfect Conductor

Within the volume of a perfect ($\sigma = \infty$) conductor, for time-varying (or static) fields, $\mathbf{E_c} = \mathbf{B_c} = 0$. The existence of surface charges and currents at the conductor's surface allows the existence of $E_\perp$ and $B_\parallel$ just outside the surface, thus rendering two of the canonical boundary conditions not useful. The two remaining (useful) conditions on the fields $\mathbf{E}$ and $\mathbf{B}$ external to the conductor, with $\mathbf{E_c} = \mathbf{B_c} = 0$, are

$$(\hat{\mathbf{n}} \times \mathbf{E})|_S = 0 \tag{16}$$

$$(\hat{\mathbf{n}} \cdot \mathbf{B})|_S = 0 \tag{17}$$

In the case of an infinitely long conductor extending symmetrically in the z-direction, $\hat{\mathbf{n}}$ is everywhere normal to $\hat{\mathbf{z}}$. Then Eqn (16) immediately implies $E_z|_S = 0$.

If, additionally, the fields are of the form (4,5), they must satisfy Maxwell's equations in the form (7-13), and Eqn (12) in particular. The boundary condition on B_z is derived from $\{(\hat{\mathbf{n}}\cdot[\text{Eqn 12}])|_S\}$ which becomes $\{ik(\hat{\mathbf{n}}\cdot\mathbf{B_t})|_S + i\frac{\omega}{c^2}\hat{\mathbf{z}}\cdot(\hat{\mathbf{n}} \times \mathbf{E_t})|_S = (\hat{\mathbf{n}} \cdot \boldsymbol{\nabla_t} B_z)|_S = 0\}$ using (16,17) and some vector manipulation. Thus the boundary condition on B_z boils down to $\frac{\partial B_z}{\partial n}|_S = 0$.

Therefore the appropriate boundary conditions on the z-components of the fields at a perfect conductor, for the z-symmetric wave problem, are

$$E_z|_S = 0 \tag{18}$$

$$\left.\frac{\partial B_z}{\partial n}\right|_S = 0 \tag{19}$$

which can be identified as the homogeneous Dirichlet and Neumann boundary conditions, respectively.

Satisfying (18,19) does not a priori satisfy the full set of boundary conditions (e.g. the transverse part). In general, all boundary conditions should be checked against a solution for consistency. However, for both TEM and TE/TM modes we obtain $\mathbf{E_t} \perp \mathbf{B_t}$. For TEM modes, making the conducting surface an equipotential in the corresponding electrostatic problem ensures all BC's are met. Presumably (14,15) together with (18,19) similarly imply that all BC's are met for the TE/TM modes.

TEM Waves

TEM waves are solutions characterized by

$$E_z = B_z = 0$$

In this case Maxwell's equations (7-10) become

$$\boldsymbol{\nabla_t} \cdot \mathbf{E_t} = 0 \qquad\qquad \boldsymbol{\nabla_t} \times \mathbf{E_t} = 0 \tag{20}$$
$$\boldsymbol{\nabla_t} \cdot \mathbf{B_t} = 0 \qquad\qquad \boldsymbol{\nabla_t} \times \mathbf{B_t} = 0 \tag{21}$$

which allows us to write

$$\mathbf{E_t} = -\boldsymbol{\nabla_t}\phi(x, y) \qquad \text{where} \qquad \nabla_t^2\phi = 0 \tag{22}$$

Using (22), the wave equation (13) implies

$$\omega = c|k| \tag{23}$$

since $\nabla_t^2\mathbf{E_t} = -\nabla_t^2\boldsymbol{\nabla_t}\phi = -\boldsymbol{\nabla_t}\nabla_t^2\phi = 0$ by interchanging partial derivatives. Then Eqn (12) implies

$$\mathbf{B_t} = \frac{\pm 1}{c}\hat{\mathbf{z}} \times \mathbf{E_t} \tag{24}$$

Notice that (23) and (24) are identical to plane waves in free space. The minus sign is for negative k values.

In summary, when TEM waves are supported, there is one TEM "mode" which supports waves at all frequencies with trivial dispersion, since $\omega = ck$. The fields are specified by the potential ϕ which satisfies $\nabla_t^2\phi = 0$ for some relevant boundary conditions. But the allowed frequencies and the dispersion relation do not depend on ϕ. TEM waves are supported when the geometry and boundary conditions admit a non-trivial solution of Laplace's equation.

Plane waves in free space are an example of TEM waves. For plane waves, the potential solves $\nabla_t^2\phi = 0$ but doesn't adhere to any physically meaningful boundary conditions and is unbounded at infinity. This corresponds do the fact that waves of truly infinite extent are clearly non-physical.

TEM Recipe

1. The dispersion relation is $\omega = c|k|$.

2. Calculate ϕ from $\nabla_t^2\phi = 0$ with electrostatic boundary conditions such that the (2D slice of) conducting surfaces are equipotential.

3. The field coefficients are given by

$$E_z = B_z = 0$$

$$\mathbf{E_t} = -\boldsymbol{\nabla_t}\phi$$

$$\mathbf{B_t} = \frac{\pm 1}{c}\hat{\mathbf{z}} \times \mathbf{E_t}$$

TM/TE Waves

TM/TE waves are solutions with longitudinal field components. Find the modes by recognizing that each field component must satisfy the wave equation (13), so in particular (13) applies to the longitudinal E_z and B_z. Fortunately, if E_z and B_z satisfy the wave equation (13) and the transverse fields are specified by (14,15), then the transverse components automatically also satisfy (13), which can be verified by interchanging partial derivatives. Note that an eigenvalue $\gamma^2 = 0$ renders (14,15) indeterminate.

Thus the problem of TM/TE waves is reduced to the problem of finding fields E_z and B_z satisfying the wave equation (13) and the boundary conditions (18,19). The dispersion relations are then dictated by the corresponding wave equation eigenvalues, and the transverse field components are determined by (14,15). The fields produced by this recipe automatically satisfy the rest of Maxwell's equations and the rest of the boundary conditions.

To simplify the calculations, we find separate solutions where either $E_z = 0$ or $B_z = 0$, called the TE and TM solutions, and linearly combine them after to get a general solution.

TM/TE Recipe

1. Solve the 2D Helmholtz equation for either TM or TE conditions, obtaining the eigenvalues γ^2 and the eigenfunctions ψ.

$$\begin{array}{cc} \text{TM} & \text{TE} \\ E_z = \psi \quad B_z = 0 & E_z = 0 \quad B_z = \psi \\ \psi|_S = 0 & \frac{\partial\psi}{\partial n}\Big|_S = 0 \end{array}$$

$$(\boldsymbol{\nabla_t}^2 + \gamma^2)\psi = 0$$

2. The dispersion relation for each eigenvalue γ^2 is

$$k^2 = \frac{\omega^2}{c^2} - \gamma^2$$

3. The transverse field coefficients for each eigenvalue are given by (14,15).

Cylindrical Cavities

Cylindrical cavities formed by putting conducting endcaps on a waveguide can be treated with the same formalism. To get cavity solutions simply form standing waves by superimposing traveling wave solutions. The extra boundary conditions at the caps then put an additional restriction on k.

In effect this means that to get the modes one just replaces the exponential factor with a sinusoidal factor and adds the new boundary conditions, then breaks the result up into traveling components to get the fields.

Finite Conductivity

To approximately find the fields near the surface of a real conductor with finite but large conductivity σ, we essentially make a first order correction to the case of the perfect conductor. For the perfect conductor we had the interior fields $\mathbf{E_c} = \mathbf{B_c} = 0$, surface charge $\mathbf{\Sigma}$, surface current $\mathbf{K}$, and thus exterior fields with $\mathbf{E}_{\parallel} = \mathbf{B}_{\perp} = 0$, with the other field components being allowed in accordance with the surface sources.