Notes on positive/negative frequency decomposition in curved spacetime scalar field theory

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These notes discuss the existence and general construction of "positive/negative frequency" mode decompositions in curved spacetime quantum field theory, in particular for the case of a Klein-Gordon scalar field. It is shown that that choosing a positive/negative-frequency basis in the vector space of complex field solutions is equivalent to choosing a symplectic basis in the vector space of real field solutions, and many such bases always exist. These results are not new—the idea is simply to provide a short, self-contained treatment of a topic often buried in technical discussion.

I. INTRODUCTION

The Hilbert space of a quantum field theory in curved spacetime is generally defined as a Fock space of solutions to the free part of the classical field equations [1–4]. In order to obtain a valid representation of the canonical commutation relations, classical modes used to define the quantum Hilbert space must form an "orthonormal, positive-frequency, complete set" of classical solutions.

Despite its name, this decomposition has little to do with frequency—it should really be called a "conjugate pair decomposition" or "symplectic decomposition." The name "positive/negative frequency decomposition" is a holdover from its most common incarnation in flat (or static) spacetime. We retain the standard name for continuity with the literature.

The purpose of these notes is to discuss positive/negative frequency decompositions in a general framework, by showing how to construct a general set of positive/negative frequency modes. This construction is not new, and has also been discussed in a number of other places, including *e.g.* [5–7].

To outline the approach, positive/negative frequency decompositions are constructed as follows. The full space of classical field solutions is a complex vector space V with involution, and with a non-degenerate (but not positive-definite) sesquilinear inner product. The real subspace V_r is a symplectic vector space. A "positive/negative-frequency" basis for V is essentially a complexified version of a symplectic basis for V_r.

This illuminates why positive/negative frequency bases are important in quantizing field theories: quantization of classical theories respects the underlying symplectic structure. (The quintessential illustration of this is the close relation $[\hat{x}_i, \hat{p}_j] = i\{x_i, p_j\} = i\delta_{ij}$ between canonical commutation relations and canonical Poisson brackets; the Poisson bracket is a symplectic form on phase space.)

In the spirit of simplicity, within these notes the symplectic structure of real solutions is derived "top-down" from within the quantum framework. Conceptually, however, it may be more correct to view the classical symplectic structure as underlying the quantum theory [5].

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II. THE VECTOR SPACE OF FREE SOLUTIONS IN SCALAR FIELD THEORY

Consider a scalar field theory defined by the action $S = S_{\text{free}} + S_{\text{int}}$, where S_{int} describes some interaction terms, and

$$S_{\text{free}} = \tfrac{1}{2} \int d^4x \sqrt{|g|}\, (\partial_\mu \phi\, \partial^\mu \phi - m^2 \phi^2).$$

The theory is defined on a fixed curved background spacetime with metric $g_{\mu\nu}$.

The free part of the action S_{free} defines a real linear field equation $L\phi = 0$. Let Ω be the space of all complex-valued classical solutions ϕ of the free field equation. Since the equation is linear, Ω forms a vector space over $\mathbb{C}$.

Since the field equation is real, complex conjugation preserves solutions, as $\phi \in \Omega$ if and only if $\phi^* \in \Omega$. Thus complex conjugation defines an $\mathbb{R}$-linear involution $* : \Omega \to \Omega$ such that $\phi^{**} = \phi$ and $(\alpha\phi)^* = \alpha^* \phi^*$ ($\alpha \in \mathbb{C}$).

A relativistically invariant complex inner product $(\phi_1, \phi_2) \in \mathbb{C}$ on free solutions may be derived from the equations of motion [1, 2]. This inner product is linear in the first argument, $(\alpha\phi_1, \phi_2) = \alpha(\phi_1, \phi_2)$ for $\alpha \in \mathbb{C}$, and has the properties $(\phi_1, \phi_2) = (\phi_2, \phi_1)^* = -(\phi_2^*, \phi_1^*)$. (Thus it is conjugate-linear in the second argument.) This inner product is non-degenerate, but not positive-definite.

In the transition to quantum field theory it is useful to decompose Ω into so-called positive and negative frequency subspaces, as follows [1–3]. In general curved spacetimes, a "complete set of orthonormal positive-frequency modes" is defined as as a set of solutions $\xi_k \in \Omega$ such that[1]

$$\phi = \sum_k \left(c_k\, \xi_k + d_k\, \xi_k^* \right) \tag{1}$$

is an arbitrary complex solution (where c_k, d_k are complex coefficients), and such that

$$(\xi_k, \xi_{k'}) = \delta_{kk'}, \qquad (\xi_k, \xi_{k'}^*) = 0. \tag{2}$$

[1] The discrete index goes over to a continuous one by standard conventions [1].

In other words, $\{\xi_k\} \cup \{\xi_k^*\}$ form an "orthonormalized" basis for Ω consisting of complex conjugate pairs. (Note that "orthonormality" here is somewhat different from in a positive-definite inner product space.) The complex spans of ξ_k and ξ_k^* are called the positive and negative frequency subspaces, respectively. There are, in general, many such decompositions.

The significance of the positive/negative frequency decomposition in quantum field theory is as follows. [[Add quantum thing.]]

In Minkowski space, one such mode decomposition is given by complex exponentials (in rectangular Minkowski coordinates) with positive/negative frequencies—hence the name. Similar sets of modes can also be found in static or asymptotically flat spacetimes, where positive-frequency modes relative to Killing vector fields have an interpretation in terms of conserved energies [2]. But in the general case considered here modes need not have any well defined frequency, and the interpretation is more abstract. The physical significance of this decomposition in general spacetimes is a somewhat complicated issue (see *e.g.* [5]). Nonetheless, the decomposition is well-defined in general, and retains an interpretation in terms of classical symplectic structure.

III. POSITIVE/NEGATIVE FREQUENCY DECOMPOSITIONS

Having extracted the relevant structural properties of the vector space of solutions, we now treat the question of positive/negative frequency decomposition as a purely linear algebraic issue. We will develop here a series of results leading to the decomposition.

A. The vector space V

Definition 1. *Let V be a vector space over $\mathbb{C}$ with an operator $*$ and a complex-valued inner product (v, w), defined as follows.*

The operator $* : V \to V$ is a conjugate-linear involution, meaning

$$\begin{gathered} v^{**} = v \\ (av_1 + bv_2)^* = a^* v_1^* + b^* v_2^* \end{gathered} \tag{3}$$

for $v \in V$ and $a, b \in \mathbb{C}$. Acting on complex numbers, $*$ is complex conjugation.

The form $(v, w) \mapsto \mathbb{C}$ is a sesquilinear inner product, linear in the first argument

$$(av_1 + bv_2, w) = a(v_1, w) + b(v_2, w) \tag{4}$$

for $v, w \in V$ and $a, b \in \mathbb{C}$, with the properties

$$(v, w) = (w, v)^* = -(w^*, v^*). \tag{5}$$

Thus it is conjugate-linear in the second argument. It is non-degenerate ($(v, w) = 0$ for all $w \in V$ only if $v = 0$), but not positive-definite.

B. The real subspace V_r

Proposition 2. *The real subspace*

$$V_r = \{v \in V \mid v^* = v\} \tag{6}$$

is a vector space over $\mathbb{R}$.

Proof. From $0^* = 0$ and $(rv)^* = rv$ for $r \in \mathbb{R}$, $v \in V_r$. □

Proposition 3. *Every element $v \in V$ can be decomposed into real and imaginary parts. That is,*

$$v = \mathrm{Re}(v) + i\,\mathrm{Im}(v) \tag{7}$$

where

$$\begin{aligned} \mathrm{Re}(v) &= \tfrac{1}{2}(v + v^*) \in V_r \\ \mathrm{Im}(v) &= \tfrac{1}{2i}(v - v^*) \in V_r. \end{aligned} \tag{8}$$

Proof. Trivial. □

Proposition 4. *Let $\mathcal{B} \subset V_r$ be a set of $\mathbb{R}$-linearly independent real vectors. Then*

$$V_r = \mathrm{span}_{\mathbb{R}}(\mathcal{B}) \iff V = \mathrm{span}_{\mathbb{C}}(\mathcal{B}). \tag{9}$$

Any basis of V_r is a basis of V.

Proof. $\mathbb{R}$-linear independence of real vectors implies $\mathbb{C}$-linear independence. Rightward implication by Prop. 3. Leftward implication since real vectors must have real coefficients in a real basis. □

For the definition of a symplectic form on a vector space, see *e.g.* [8].

Proposition 5. *The form*

$$\langle v, w\rangle = -i(v, w) \tag{10}$$

on V_r is real-valued and obeys:

(i) $\mathbb{R}$-Linearity in both arguments.

(ii) Antisymmetry: $\langle v, w\rangle = -\langle w, v\rangle$.

(iii) Non-degeneracy.

Thus V_r is a real symplectic vector space with symplectic form $\langle v, w\rangle \mapsto \mathbb{R}$.

Proof. Let $v, w \in V_r$ and $r, s \in \mathbb{R}$. It is real-valued since $(-i(v, w))^* = i(v, w)^* = -i(v^*, w^*) = -i(v, w)$. Antisymmetry derives as $\langle v, w\rangle = -i(v, w) = i(w^*, v^*) = -\langle w, v\rangle$. Linearity follows from $(v, rw) = r^*(v, w) = r(v, w)$ with the properties of (v, w). Non-degeneracy is by definition of (v, w). □

Definition 6. *A symplectic basis for V_r is a basis*

$$\mathcal{B} = \{e_m\} \cup \{f_m\} \tag{11}$$

such that

$$\begin{aligned} \langle e_m, e_n \rangle = \langle f_m, f_n \rangle = 0, \\ \langle e_m, f_n \rangle = \delta_{mn}. \end{aligned} \tag{12}$$

Note that the basis elements come in pairs. (The indices $m, n \in M$ have the same index set for both e_m and f_m.)

Theorem 7. *Given a linearly independent set $\{e_m\} \subset V_r$ such that $\langle e_m, e_n \rangle = 0$, there exists a symplectic basis $\mathcal{B}$ containing $\{e_m\}$.*

Proof. Thm. 1.15 of [8]. □

Proposition 8. *Every real vector $v \in V_r$ is orthogonal to itself, $\langle v, v \rangle = 0$. Therefore, given any real vector $v \in V_r$, there exists a symplectic basis for V_r containing it.*

Proof. Antisymmetry and Thm. 7. □

C. The decomposition

Definition 9. *If $\mathcal{B} \subset V_r$ is a symplectic basis for V_r, then by Prop. 4 it is also a basis for V, which we call a symplectic basis for V.*

Theorem 10. *Let $\mathcal{B}_0 = \{e_m\} \cup \{f_m\}$ be a symplectic basis for V, and let*

$$g_m^\pm = \tfrac{1}{\sqrt{2}}(e_m \pm i\, f_m). \tag{13}$$

Then $g_m^- = (g_m^+)^*$, and $\mathcal{B} = \{g_m^+\} \cup \{g_m^-\}$ is a basis for V such that

$$\begin{aligned} (g_m^+, g_n^+) &= -(g_m^-, g_n^-) = \delta_{mn} \\ (g_m^+, g_n^-) &= (g_m^-, g_n^+) = 0. \end{aligned} \tag{14}$$

We call $\mathcal{B}$ a positive/negative-frequency basis for V. The spans of $\{g_m^+\}$ and $\{g_m^-\}$ are called, respectively, the positive- and negative-frequency sectors.

Proof. Direct calculation using $-i(v_r, w_r) = \langle v_r, w_r \rangle$ for real vectors $v_r, w_r \in V_r$. Linear independence of $\mathcal{B}$ follows from $\mathcal{B}_0$. □

IV. CONCLUSIONS

A positive/negative frequency basis in the vector space of complex classical field solutions is a complexified version of a symplectic basis in the vector space of real field solutions. There exists a symplectic basis containing any given real solution.

In general curved spacetimes there is no way to single out which decomposition into positive and negative frequency modes is physically relevant—indeed, different decompositions may be relevant to different observers, as is the case, for example, in the Unruh effect [9].

An interesting result of Ashtekar and Magnon [5], however, is that a 3+1 splitting of spacetime *does* in fact determine a unique (but time-dependent) choice of positive/negative-frequency sectors. From this perspective, a description of physics in terms of "particles" is one of the many concepts of non-relativistic physics that make sense only when a global splitting of space and time is agreed upon.

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