The multivector approach to differential geometry: a simpler foundation for general relativity

Joe Schindler
UCSC Physics

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Overview

How can Clifford algebra
(aka "multivector algebra" or "geometric algebra")
be applied to simplify GR?

WARNING: This is a math talk. It is about the *mathematical foundations* of GR, not about GR itself.

Based on arxiv:1911.07145 in math.DG.

J. Schindler. *Geometric Manifolds Part I: The Directional Derivative of Scalar, Vector, Multivector, and Tensor Fields.* 2019.

Contents

Introduction to Clifford algebra

Clifford's "Geometric algebra"

- Clifford himself called his algebra *Geometric Algebra* (GA).
- Familiar as the algebra of σ_i and γ_μ, which are matrix representations of $GA(3)$ and $GA(3,1)$ respectively.
- But, more useful to do GA *without* matrix rep.

GA is much more broadly applicable than often recognized: it is the algebra of vectors in physical space.

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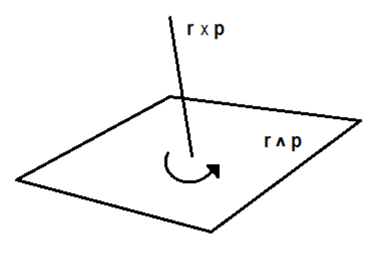
Motivation: A simple question

How do you define the angular momentum $L = \vec{r} \times \vec{p}$ in N dimensions?



L **is a plane, not a (pseudo-)vector!**

Only in 3d a plane defines a unique normal vector.

Motivation: A simple question II

Similarly, what is the nature of the magnetic field?

2+1d	4+1d
$F = \begin{pmatrix} 0 & E_1 & E_2 \\ & 0 & B_1 \\ & & 0 \end{pmatrix}$	$F = \begin{pmatrix} 0 & E_1 & E_2 & E_3 & E_4 \\ & 0 & B_1 & B_2 & B_3 \\ & & 0 & B_4 & B_5 \\ & & & 0 & B_6 \\ & & & & 0 \end{pmatrix}$
B has 1 component	B has 6 components

B **is a bivector, NOT a pseudovector!**

Only in 3d are these equivalent.

So what is GA?

A **Geometric Algebra** is a linear space incorporating vectors with both *dot* and *wedge* multiplication.

It is built by using a wedge product to extend a vector space with inner product. It contains exterior algebra as a subalgebra.

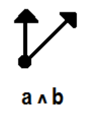
Its elements are called **multivectors**. These represent oriented lengths, areas, and volumes, and can be visualized much like vectors.

So what is GA?

Start with a vector space with inner product over $\mathbb{R}$.

The dot product $a \cdot b$ is a scalar.

The wedge product $a \wedge b$



is a new type of object called a 2-vector (or "bivector").

So what is GA?

The dot and wedge have all the properties you expect.

Most importantly:

- $a \cdot b = b \cdot a$
- $a \wedge b = -\, b \wedge a$
- $a \wedge b \wedge c = (a \wedge b) \wedge c = a \wedge (b \wedge c)$

These imply wedge antisymmetric on all swaps.

So what is GA?

$GA(n,m)$
Built on $\mathbb{R}(n,m)$

$GA(2) =$ algebra of the plane
Built on 2d Euclidean space

$GA(3) =$ algebra of space
Built on 3d Euclidean space

$GA(3,1) =$ algebra of spacetime
Built on 3+1d Minkowski space

GA(3)

$GA(N)$ is real linear space of all k-vectors in N-dimensional space.

Grade	$GA(3)$ Basis Elements	Name
0	1	scalar
1	e_1, e_2, e_3	vector
2	$e_1 \wedge e_2$, $e_2 \wedge e_3$, $e_3 \wedge e_1$	bivector/pseudovector
3	$e_1 \wedge e_2 \wedge e_3$	trivector/pseudoscalar

Oriented lengths, areas, and volumes.

GA(2)

Compare $GA(3)$ to $GA(2)$.

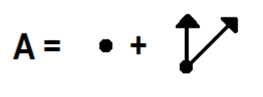
Grade	$GA(2)$ Basis Elements		Name
0	1		scalar
1	e_1	e_2	vector/pseudovector
2	$e_1 \wedge e_2$		bivector/pseudoscalar

MULTIVECTORS

A multivector A is a linear combination of k-vectors.

$$A = \sum_k \langle A \rangle_k$$

Different grades are linearly independent—visualize a formal sum of various k-vectors.



Geometric Multiplication

Fundamental Identity of GA.

The geometric product of vectors a, b defined by

$$ab = a \cdot b + a \wedge b$$

is a scalar plus a bivector.

Associative, invertible*, non-commutative.

Implies

$$\begin{aligned} a \cdot b &= (ab + ba)/2 \\ a \wedge b &= (ab - ba)/2. \end{aligned}$$

This combining of grades is the key step making GA extremely powerful—more powerful than tensors or differential forms.

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Geometric Multiplication

That was for vectors. What about multivectors?

Geometric product AB is fundamental.

Dot and wedge

$$\begin{aligned} A_j \cdot B_k &= \langle AB \rangle_{k-j} \\ A_j \wedge B_k &= \langle AB \rangle_{k+j} \end{aligned}$$

are maximally grade raising and lowering parts (extended to general A, B by linearity).

Geometric Multiplication

For particle physicists' eyes only:

Matrix Rep	GA
$\gamma^\mu\gamma^\nu = g^{\mu\nu} + i\sigma^{\mu\nu}$	$ab = a \cdot b + a \wedge b$
$\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu}$	$\frac{1}{2}(ab + ba) = a \cdot b$
$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$	$I = e_0 \wedge e_1 \wedge e_2 \wedge e_3$
$\{\gamma^5, \gamma^\mu\} = 0$	$I \wedge a = 0$
$\partial\!\!\!/ = \gamma^\mu \partial_\mu$	$\boldsymbol{\nabla} = e^i\, \nabla_{e_i}$

In flat space a **gradient operator** on multivector fields can be defined in the form

$$\nabla A = e^i \, \partial_i \, A$$

such that (using $ab = a \cdot b + a \wedge b$)

$$\begin{aligned} \nabla A &= e^i \cdot \partial_i \, A + e^i \wedge \partial_i \, A \\ &= \nabla \cdot A + \nabla \wedge A \end{aligned}$$

generalizing the gradient, divergence, and curl of vector calculus.

Geometric Calculus

There is a Fundamental Theorem of Geometric Calculus like

$$\int_M \nabla F = \int_{\partial M} F$$

generalizing Stokes' theorem of differential forms, and the curl and divergence theorems of vector calculus.

- Strictly more useful than Stokes', because antiderivative exists more generally. No restriction to exact forms on M.
- LHS integrand has simple interpretation as gradient, unlike d.

Why is this framework so useful?

Applications

Direct applications to:

- Rotational motion
- Electrodynamics
- Spinors
- Complex analysis
- Lie algebras
- ...

Electrodynamics

Lagrangian:

Tensor calculus:	$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + A^\mu J_\mu$	$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$
Differential forms:	$\mathcal{L} = -\frac{1}{2} F \wedge *F + A \wedge *J$	$F = dA$
Geometric calculus:	$\mathcal{L} = -\frac{1}{2} F \cdot F + A \cdot J$	$F = \boldsymbol{\nabla} \wedge A$.

Equations of motion:

Tensor calculus:	$\nabla_\mu F^{\mu\nu} = J^\nu$	$\epsilon^{\alpha\beta\mu\nu} \nabla_\alpha F_{\mu\nu} = 0$
Differential forms:	$d * F = *J$	$dF = 0$
Geometric calculus:	$\boldsymbol{\nabla} \cdot F = J$	$\boldsymbol{\nabla} \wedge F = 0$.

Only GA concisely expresses both metric-compatible divergence and metric-independent exterior derivative. Note metric info is in Hodge star.

Electrodynamics

Only in GA these combine to one equation

$$\boldsymbol{\nabla} F = J$$

which splits into vector and trivector parts

$$\begin{array}{ccccccc} \boldsymbol{\nabla} \cdot F & + & \boldsymbol{\nabla} \wedge F & = & J & + & 0 \\ (1) & + & (3) & = & (1) & + & (3). \end{array}$$

Not just notational improvement!

Unlike others, GA ∇ is invertible by integral operator so that

$$F = \nabla^{-1} J = \int dx'\, G(x, x') J(x')$$

where G is some Green's function. Relativistic E&M without A!

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Spinors

In GA a **spinor** is an even-grade multivector.

$$GA(3) \Longleftrightarrow 4 \text{ real dof's} \Longleftrightarrow 2\text{-component complex spinor}$$

$$GA(3,1) \Longleftrightarrow 8 \text{ real dof's} \Longleftrightarrow 4\text{-component complex spinor}$$

Dirac's equation in this context reads (with I unit pseudoscalar)

$$\boldsymbol{\nabla}\psi = m\psi I.$$

Maxwell's and Dirac's equations are both equalities of multivectors, and use the same derivative!

Complex analysis

In $GA(2)$ write the unit pseudoscalar as $i = e_1 \wedge e_2$.

Note $i^2 = -1$.

A complex number is $z = x + iy$ for scalars x, y.

Then $\boldsymbol{\nabla} F = 0$ is equivalent to the Cauchy-Riemann equations.

Fund Thm of Calc $\Longrightarrow$ Residue Thm + Cauchy Integral Thm

Generalizes complex analysis to N dimensions.

Bivectors

A simple unit bivector B specifies a **plane** (e.g. $B = e_1 \wedge e_2$).

Since $B^2 = -1$ we have for scalar θ

$$e^{B\theta} = \cos\theta + B\sin\theta.$$

This generates rotations in the plane of B by

$$A \to e^{-B\theta/2}\ A\ e^{B\theta/2}.$$

Bivectors

The set of bivectors in $GA(n,m)$ forms the Lie algebra

$$\mathfrak{so}(n,m)$$

under commutator product

$$[B,B'] = BB' - B'B.$$

Every Lie algebra is isomorphic to a bivector Lie algebra (since one can realize $\mathfrak{gl}(n,\mathbb{R})$ within $GA(n,n)$).

Bivector algebras naturally define a Lie group action on vectors.

And...

Also includes Grassmann algebra, quaternions, and more!

Takeaway

GA is a natural framework for many parts of mathematical physics!

Differential geometry

How can we use GA methods in the context of GR?
Need formalism for GA on smooth manifolds.

In standard Riemannian geometry one avoids introducing the metric until the last possible moment, for maximum "generality".

But including the metric from the beginning and using GA leads to much more powerful theory.

Hopefully I can convey how...

Differential geometry

How can we use GA methods in the context of GR?
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In standard Riemannian geometry one avoids introducing the metric until the last possible moment, for maximum "generality".

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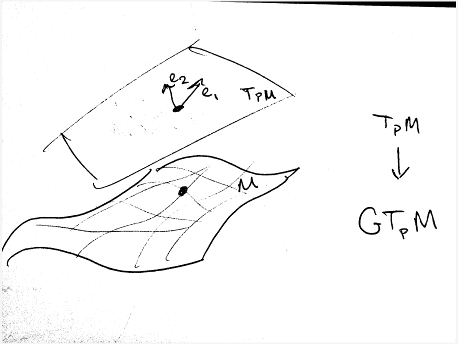
Proceed as follows

First I'll set up some basic formalism.

- Geometric manifold
- Reciprocal bases

Then highlight some of the main new results, comparing to standard methods.

A **geometric manifold** is a smooth manifold with a metric.



The tangent space is extended to a **geometric tangent space** of "tangent multivectors" using the metric.

Reciprocal bases

Many simplifications come from using **pairs of reciprocal bases**.

Given a vector basis e_i with metric coefficients

$$e_i \cdot e_j = g_{ij}$$

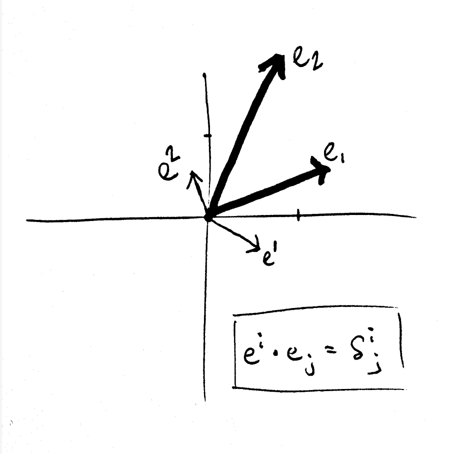
there exists a reciprocal basis e^i defined by

$$e^i \cdot e_j = \delta^i_j.$$

It follows that $e^i = g^{ij} e_j$.

Reciprocal bases

The basis and reciprocal basis live in the same vector space!



There is no need (or reason) to introduce the dual tangent space.

Reciprocal bases

Arbitrary vectors can be decomposed as

$$a = (a \cdot e^i)\, e_i = a^i\, e_i$$

or

$$a = (a \cdot e_i)\, e^i = a_i\, e^i.$$

Thus a_i and a^i are *reciprocal components* for the same vector a.

These are like covariant and contravariant components, but without the technical distinction between vectors and covectors.

New notation

Directional derivative of scalar field φ in direction of vector a.

Usually would write $a\varphi$ (or more commonly $X\varphi$).

Instead, we write $\partial_a\varphi$ for the same derivative.

The "derivative of φ in the direction a".
The direction argument is a vector, *never an index*.

Special bases

Coordinate basis written $e(x_i)$ such that

$$\partial_{e(x_i)}\varphi = \frac{\partial\varphi}{\partial x^i} \ .$$

Coordinate gradient basis

$$dx^i = \boldsymbol{\nabla} \wedge x^i = \boldsymbol{\nabla} x^i = g^{ij} e(x_j)$$

is defined as the gradients of the scalar coordinate functions.

These are reciprocal

$$e(x_i) \cdot dx^j = \delta_i^j$$

and both are vectors living in the same space.

Special bases

Moreover with two coordinate systems

$$e(x_i) \cdot dy^j = \frac{\partial y^j}{\partial x^i}$$

leading to standard change of coordinate formulae.

Special bases

In summary.

Special bases:

	Arbitrary	Orthonormal	Holonomic	Gradient
Definition:	e_i	$g_{ij} = \eta(i)\, \delta_{ij}$	$L_{ijk} = 0$	$e_i = \nabla \varphi_i$
Reciprocal to:	$e^i = g^{ij}\, e_j$	*Orthonormal*	*Gradient*	*Holonomic*

Coordinate bases:

Coordinate Basis	Coordinate Gradient Basis
$e(x_i)$	$dx^i = \nabla x^i = g^{ij}\, e(x_j)$
Type: Holonomic	*Type: Gradient*

Results

Now let's look at some of the results.

Connection

Affine connection relates neighboring tangent spaces.

An **arbitrary basis** has metric

$$e_i \cdot e_j = g_{ij}$$

and Lie bracket coefficients

$$[e_i, e_j] = L_{ijk}\, e^k.$$

Let D be an affine connection with connection coefficients

$$D_{e_i} e_j = \Gamma_{ijk}\, e^k.$$

Connection

Metric Compatible if and only if:

$$\Gamma_{ijk} + \Gamma_{ikj} = \partial_{e_i} g_{jk}$$

Torsion Free if and only if:

$$\Gamma_{ijk} - \Gamma_{jik} = L_{ijk}$$

This leads to

$\Gamma_{ijk} = \frac{1}{2}\left(\partial_{e_i} g_{jk} - \partial_{e_k} g_{ij} + \partial_{e_j} g_{ki}\right) + \frac{1}{2}\left(L_{ijk} - L_{jki} + L_{kij}\right) + \chi_{ijk}$

where $\chi_{ijk} = 0$ (zero **contorsion coefficients**) gives Levi-Civita connection.

This reduces to standard form for holonomic basis, and reduces to Cartan's spin connection for orthonormal basis. Surprisingly rare formula.

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Multivector directional derivative

We know how to take derivative of scalar and vector fields.
What about multivector fields?

Multivector directional derivative

Definition 8 (Multivector directional derivative). Let A and B be smooth multivector fields, let a and b be vectors based at a point p, and let α and β be scalars at p. Let $D : T_pM \times MVF(M) \to GT_pM$ be an operator mapping a tangent vector a at p and a smooth multivector field A in a neighborhood of p to a tangent multivector D_aA at p. D is called a *multivector directional derivative* (MDD) if it has the properties:

(i) $D_{(\alpha a+\beta b)}A = \alpha D_aA + \beta D_bA,$ (linearity in the direction argument)

(ii) $D_a\langle A\rangle_0 = \partial_a\langle A\rangle_0\,,$ (scalar-field directional derivative on scalars)

(iii) $D_a\langle A\rangle_1 = \langle D_a\langle A\rangle_1\rangle_1\,,$ (preserves grade of vectors)

(iv) $D_a(A+B) = D_aA + D_aB,$ (linearity in the field argument)

(v) $D_a(AB) = (D_aA)B + A(D_aB).$ (product rule)

When D is a multivector directional derivative, D_aA is called the derivative of the multivector field A in the direction a at the point p.

MAIN RESULT: Such MDD operators exist, and are in bijective correspondence with metric-compatible affine connections. Far from trivial, due to product rule requirement.

Multivector directional derivative

This leads to an extremely powerful notion of directional derivative which

- Is grade-preserving.
- Is Leibniz on AB and $A \cdot B$ and $A \wedge B$.
- Acts as a metric compatible connection on vectors.
- Is fully specified by a set of connection coefficients.

The unique torsion free MDD is called ∇.

Grad Div Curl

The torsion-free gradient is

$$\boldsymbol{\nabla} A = e^i \nabla_{e_i} A$$

where A is any multivector field.

Theorem 29 (Gradient = divergence + curl). Let D be a multivector directional derivative, let e_i be an arbitrary basis, and let A be a multivector field. Define the *divergence* and *curl* by

$$\boldsymbol{D} \cdot A = e^i \cdot D_{e_i} A, \qquad \text{(divergence)}$$

$$\boldsymbol{D} \wedge A = e^i \wedge D_{e_i} A. \qquad \text{(curl)}$$

Then the gradient equals the divergence plus the curl,

$$\boldsymbol{D} A = \boldsymbol{D} \cdot A + \boldsymbol{D} \wedge A. \tag{39}$$

These definitions are consistent with the usual definitions of dot and wedge product, and can be shown to be basis-independent and thus well-defined.

These reduce to the standard grad div and curl in flat space!

Differential forms

The theory fully includes (and clarifies the meaning of) differential forms.

The torsion-free curl

$$d = \boldsymbol{\nabla}\wedge$$

is completely equivalent to the exterior derivative of differential forms.

Differential forms

The exterior derivative has the properties

Theorem 32 (Exterior derivative properties). The exterior derivative $d = \boldsymbol{\nabla}\wedge$ of multivectors has the following properties:

(i) $d(A + B) = dA + dB$.

(ii) If φ is a scalar field then $a \cdot d\varphi = \partial_a \varphi$ for all vector fields a.

(iii) If φ is a scalar field then $d^2\varphi \equiv d(d\varphi) = 0$.

(iv) If A_j and B_k are multivectors of fixed grades j and k respectively, then

$$d(A_j \wedge B_k) = d(A_j) \wedge B_k + (-1)^j A_j \wedge d(B_k)\,.$$

(which imply $\boldsymbol{\nabla} \wedge \boldsymbol{\nabla} \wedge A = 0$ and $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \cdot A = 0$.)

If that doesn't satisfy, you can also write down an explicit isomorphism between the multivectors and forms.

Tensors

What about tensors?

Tensors

We all know that

$$D_i T^{jk} = \partial_i T^{jk} + (\Gamma_{ilm}\, g^{mj})\, T^{lk} + (\Gamma_{ilm}\, g^{mk})\, T^{jl}$$

But where does that come from?

That is, how is the connection extended from vectors to tensors?

Tensors

Wald's way is typical of any textbook:

1. Linearity: For all $A, B \in \mathscr{T}(k,l)$ and $\alpha, \beta \in R$,

$$\nabla_c(\alpha A^{a_1\cdots a_k}{}_{b_1\cdots b_l} + \beta B^{a_1\cdots a_k}{}_{b_1\cdots b_l}) = \alpha \nabla_c A^{a_1\cdots a_k}{}_{b_1\cdots b_l} + \beta \nabla_c B^{a_1\cdots a_k}{}_{b_1\cdots b_l}$$

2. Leibnitz rule: For all $A \in \mathscr{T}(k,l)$, $B \in \mathscr{T}(k',l')$,

$$\nabla_e[A^{a_1\cdots a_k}{}_{b_1\cdots b_l} B^{c_1\cdots c_{k'}}{}_{d_1\cdots d_{l'}}] = [\nabla_e A^{a_1\cdots a_k}{}_{b_1\cdots b_l}] B^{c_1\cdots c_{k'}}{}_{d_1\cdots d_{l'}} + A^{a_1\cdots a_k}{}_{b_1\cdots b_l}[\nabla_e B^{c_1\cdots c_{k'}}{}_{d_1\cdots d_{l'}}] \quad .$$

3. Commutativity with contraction: For all $A \in \mathscr{T}(k,l)$,

$$\nabla_d(A^{a_1\cdots c\cdots a_k}{}_{b_1\cdots c\cdots b_l}) = \nabla_d A^{a_1\cdots c\cdots a_k}{}_{b_1\cdots c\cdots b_l}$$

4. Consistency with the notion of tangent vectors as directional derivatives on scalar fields: For all $f \in \mathscr{F}$ and all $t^a \in V_p$

$$t(f) = t^a \nabla_a f \quad .$$

But the truth may shock you!
It really follows from the chain rule.

Tensors

A tensor $T(A, B)$ is a linear function of multivectors. So first...

Consider a real function $f(x, g(x), h(x))$ of a real variable.

As x varies, the total differential is

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial g}\, dg + \frac{\partial f}{\partial h}\, dh \; .$$

If f is linear in the g and h arguments then

$$\begin{aligned} \frac{\partial f}{\partial g}\, dg &= f(x, g + dg, h) - f(x, g, h) \\ &= f(x, dg, h) \end{aligned}$$

and likewise for the h term.

Tensors

Thus

$$df = \frac{\partial f}{\partial x}\, dx + f(x, dg, h) + f(x, g, dh)\ .$$

Only partial term is change in f itself.

For tensors this becomes

$$\begin{array}{ccccccc} df & = & \frac{\partial f}{\partial x}\, dx & + & f(x, dg, h) & + & f(x, g, dh) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ D_a(T(A, \dots, B)) & = & (DT)(a, A, \dots, B) & + & T(D_a A, \dots, B) & + \dots + & T(A, \dots, D_a B). \end{array}$$

Leading to the **tensor derivative** definition based on the chain rule

Definition 25 (Tensor derivative). Let T be a tensor field of signature $(k_1, \ldots, k_N : k_0)$, and let D be a multivector directional derivative. Define DT by

$$(DT)(a, A, \ldots, B) = D_a(T(A, \ldots, B)) - T(D_a A, \ldots, B) - \ldots - T(A, \ldots, D_a B)\,. \qquad (30)$$

DT is a tensor field of signature $(1, k_1, \ldots, k_N : k_0)$, called the *tensor derivative* of T.

This is proved equivalent to the usual version, and can easily be translated back to the usual components in a basis.

In Conclusion

This formalism provides a whole new way of doing calculus which:

- Unifies vector calculus in curved and flat spaces.
- Strictly includes the theories of tensors and differential forms.
- Includes the tetrad and spin connection formalisms.
- Admits simpler calculations.
- Works in an arbitrary basis.
- Fixes "conceptually right but technically wrong" statements in standard formalism.
- Will allow improved treatments of integral calculus and of curvature and related topics.

Future steps

Future steps:

- Define integral calculus on GM.
- Derive Riemann curvature etc on GM.

Thanks!

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