Vacuum polarization in Schwarzschild spacetime

P. Candelas
Center for Theoretical Physics, University of Texas at Austin, Austin, Texas 78712
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The polarization of the vacuum induced by gravitation is studied for massless fields in the region exterior to the horizon of a Schwarzschild black hole. The renormalized value of $\langle\phi^2(x)\rangle$ is calculated according to the "covariant point-separation scheme" for each of the Boulware, Hartle-Hawking, and Unruh "vacua." The form of the renormalized expectation value of the stress tensor near the horizon and at infinity is discussed for each of these three states. It is found that the Unruh vacuum best approximates the state that would obtain following the gravitational collapse of a massive body in the sense that the expectation values of physical observables are finite, in a freely falling frame, on the future horizon and that this state is empty near infinity apart from an outgoing flux of a blackbody radiation. The response of an Unruh box is examined further in the light of the results obtained for the stress tensor. Finally it is shown by explicit solution of the linearized Einstein equations that the area of the horizon decreases at the rate expected from the flux at infinity.

I. INTRODUCTION

In this paper our aim is to discuss the physical meaning of, and relationship between, the different measures of vacuum activity in Schwarzschild spacetime.

A measure that receives considerable attention is the renormalized vacuum expectation value of the stress-energy tensor $\langle T_\mu{}^\nu(x)\rangle_{\rm ren}$ not least because this quantity determines the evolution of the geometry in a self-consistent field approximation. Another measure that is of considerable interest is the response of the idealized particle detector proposed by Unruh.[1] The study of physical observables, of which these two are important examples, provides a means of examining the physical content of any state that might be proposed as a suitable candidate for the vacuum. Three such states have been proposed for a spherically symmetric black-hole spacetime. In order to discuss these and also introduce our conventions let us briefly review some of the relevant geometrical properties of the manifold.[2]

The metric of Schwarzschild spacetime can be given in terms of Schwarzschild coordinates in the form

$$ds^2=-\left(1-\frac{2M}{r}\right)dt^2+\frac{dr^2}{1-2M/r}+r^2(d\theta^2+\sin^2\theta d\phi^2)\,. \qquad (1)$$

These coordinates can be taken to cover the exterior region $r>2M$ of the spacetime (region I in Fig. 1). The metric (1) has a coordinate singularity at the horizon $r=2M$. This singularity may be removed by transforming to nonsingular coordinates such as those introduced by Kruskal which, for $r>2M$, are related to Schwarzschild coordinates by

$$\begin{aligned} v&=\left(\frac{r}{2M}-1\right)^{1/2}e^{r/4M}\sinh(t/4M)\,,\\ u&=\left(\frac{r}{2M}-1\right)^{1/2}e^{r/4M}\cosh(t/4M)\,. \end{aligned} \qquad (2)$$

In terms of these coordinates the line element becomes

$$ds^2=\frac{32M^3}{r}e^{-r/2M}(-dv^2+du^2)+r^2(d\theta^2+\sin^2\theta d\phi^2) \qquad (3)$$

in which r is understood as a function of u and v given implicitly by

$$\left(\frac{r}{2M}-1\right)e^{r/2M}=u^2-v^2\,.$$

The metric (3) is singular only at the curvature singularities where $r=0$, and with the coordinate ranges $-\infty<v<\infty$, $-\infty<u<\infty$, $v^2-u^2<1$ represents

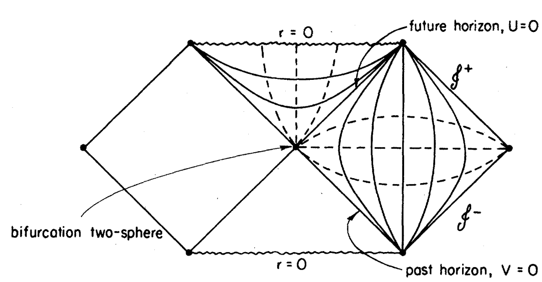

FIG. 1. The Penrose diagram for the maximally extended Schwarzschild manifold. The (θ,ϕ) coordinates have been suppressed so that, apart from the curvature singularities where $r=0$, each point represents a two-sphere. The continuous lines are the curves r =constant which are the orbits of the Killing vector $\partial/\partial t$, and the dashed lines are curves of constant t.

the maximal analytic extension of the Schwarzschild manifold. In addition to the coordinates (2) we shall have occasion to refer to Kruskal null coordinates U and V defined by

$$U = v - u\,, \qquad V = v + u\,. \tag{4}$$

With these conventions disposed of we return to the three presumptive vacua for a quantum field theory on the maximally extended Schwarzschild manifold. These are

(i) the Boulware[3] vacuum $|B\rangle$, defined by requiring normal modes to be positive frequency with respect to the Killing vector $\partial/\partial t$ with respect to which the exterior region is static,

(ii) the Unruh[1] vacuum $|U\rangle$, defined by taking modes that are incoming from $\mathscr{I}^-$ to be positive frequency with respect to $\partial/\partial t$, while those that emanate from the past horizon are taken to be positive frequency with respect to U, the canonical affine parameter on the past horizon,

(iii) the Hartle-Hawking[4] vacuum $|H\rangle$, defined by taking incoming modes to be positive frequency with respect to V, the canonical affine parameter on the future horizon, and outgoing modes to be positive frequency with respect to U.

The problem of determining the expectation value of a physical observable factors, roughly speaking, into two parts: how to implement a renormalization scheme and how to evaluate mode sums. It should be emphasized that these are distinct problems in principle, though they are often interrelated in practice since asymptotic expansions become necessary at some stage *faute de mieux*.

Largely as a preparatory exercise we consider in Sec. II the problem of renormalizing the quadratically divergent quantity $\langle\phi^2(x)\rangle$ for a massless scalar field $\phi(x)$. It turns out that it is a straightforward matter to carry out this renormalization by the geodesic point-separation technique of DeWitt[5] and Christensen[6] and obtain an explicit mode-sum representation for $\langle\phi^2(x)\rangle_{\rm ren}$.[7] We derive also, in this section, a representation for the Hartle-Hawking propagator on the Euclidean section which proves useful for studying the properties of the Hartle-Hawking vacuum near the horizon. Sections III and IV are devoted to a study of the asymptotic values of the renormalized vacuum expectation values of the stress tensor near the horizon and at infinity for each of the three vacua.

It is possible to extract a certain amount of information without performing an explicit renormalization. We may, for example, take advantage of the fact that we expect $\langle H|T_\mu{}^\nu|H\rangle_{\rm ren}$ to be regular in a freely falling frame on the horizon in order to compute the leading behavior of $\langle T_\mu{}^\nu\rangle_{\rm ren}$ in the Boulware vacuum since this quantity diverges as $r\to 2M$. Thus

$$\langle B|T_\mu{}^\nu|B\rangle_{\rm ren} \underset{r\to 2M}{\sim} \langle B|T_\mu{}^\nu|B\rangle_{\rm ren} - \langle H|T_\mu{}^\nu|H\rangle_{\rm ren} = \langle B|T_\mu{}^\nu|B\rangle - \langle H|T_\mu{}^\nu|H\rangle\,.$$

Similarly we may calculate $\langle T_\mu{}^\nu\rangle_{\rm ren}$ for the Hartle-Hawking and Unruh vacua as $r\to\infty$, taking advantage of the fact that we expect $\langle B|T_\mu{}^\nu|B\rangle_{\rm ren}$ to tend rapidly to zero there. Thus

$$\langle H|T_\mu{}^\nu|H\rangle_{\rm ren} \underset{r\to\infty}{\sim} \langle H|T_\mu{}^\nu|H\rangle_{\rm ren} - \langle B|T_\mu{}^\nu|B\rangle_{\rm ren} = \langle H|T_\mu{}^\nu|H\rangle - \langle B|T_\mu{}^\nu|B\rangle$$

and analogously for $\langle U|T_\mu{}^\nu|U\rangle_{\rm ren}$.

By introducing uniform asymptotic approximations for the mode sums involved we are able to evaluate these differences of stress tensors and establish many of the conjectures made by Christensen and Fulling[8] in their interesting paper. In Sec. IV we address the problem of explicitly renormalizing the stress tensor near the horizon by geodesic point separation. This is most easily effected for points on the bifurcation two-sphere, the two-sphere where the future and past horizons intersect. We partially extend our result to the rest of the horizon making use of the symmetry of the manifold. Unfortunately, one component of $\langle T_\mu{}^\nu\rangle_{\rm ren}$ remains undetermined by this process. We hope to return to the calculation of this remaining component in a future publication.

Our results, however, are sufficient to substantiate the following interpretation:

(i) The Boulware vacuum corresponds to our familiar concept of an empty state for large radii, but is pathological at the horizon in the sense that the expectation value of the stress tensor, evaluated in a freely falling frame, diverges as $r\to 2M$. For a massless scalar field the leading behavior of $\langle B|T_\mu{}^\nu|B\rangle_{\rm ren}$ near the horizon is given in Schwarzschild coordinates by

$$\langle B|T_\mu{}^\nu|B\rangle_{\rm ren} \sim -\frac{1}{2\pi^2(1-2M/r)^2} \times \int_0^\infty \frac{d\omega\,\omega^3}{e^{8\pi M\omega}-1}\begin{bmatrix}-1 & 0 & 0 & 0\\ 0 & \frac{1}{3} & 0 & 0\\ 0 & 0 & \frac{1}{3} & 0\\ 0 & 0 & 0 & \frac{1}{3}\end{bmatrix},$$

which corresponds to the absence from the vacuum of blackbody radiation at the black-hole temperature $(8\pi M)^{-1}$. This result is in precise analogy to the result for $\langle T_\mu{}^\nu\rangle_{\rm ren}$ above an infinitely accelerated plane "conductor" in Minkowski space.[9]

(ii) In the Unruh vacuum we find that $\langle T_\mu{}^\nu\rangle_{\rm ren}$ is regular, in a freely falling frame, on the future horizon but not on the past horizon. At infinity this vacuum corresponds to an outgoing flux of blackbody radiation at the black-hole temperature.

(iii) The Hartle-Hawking vacuum does not correspond to our usual notion of a vacuum. $\langle T_\mu{}^\nu\rangle_{\text{ren}}$ is well behaved, in a freely falling frame, on both the future and past horizons but the price paid for this is that the state is not empty at infinity, corresponding instead to a thermal distribution of quanta at the black-hole temperature. That is, the Hartle-Hawking "vacuum" corresponds to a black hole in (unstable) equilibrium with an infinite sea of blackbody radiation. We conclude from this that it is the Unruh vacuum that best approximates the state that would obtain following the gravitational collapse of a massive body.

We consider in Sec. V a different measure of vacuum activity, the response of an idealized "particle detector" of the type proposed by Unruh. We shall argue that Unruh's box does not so much measure "particles" as the spectrum of fluctuations of the quantum field, and that the interpretation of the vacuum entailed by the readings of this device is in no way incompatible with the one that follows from a consideration of, say, $\langle T_{\mu\nu}\rangle_{\text{ren}}$. It is simply the case that these two observables are largely independent measures of vacuum activity.

Finally, in Sec. VI we show, by explicit solution of the linearized Einstein equations, that in the Unruh vacuum the area of the black hole decreases at the rate expected from the magnitude of the flux at infinity. This result is, of course, completely obvious on physical grounds but the fact that it may be derived by solving the Einstein equations may be regarded as a tentative step towards the solution of the back-reaction problem.

The picture that emerges from these considerations is that a consistent view of vacuum activity and particle production is possible in black-hole spacetimes, and that, in particular, there is no infinite energy density associated with the production of Hawking radiation near the horizon in either the Unruh or Hartle-Hawking vacuum.

II. THE RENORMALIZATION OF $\langle\phi^2(x)\rangle$

As a measure preparatory to the renormalization of $\langle T_\mu{}^\nu\rangle$ we consider here the somewhat simpler, though related, problem of renormalizing the quadratically divergent $\langle \phi^2(x)\rangle$ for a massless scalar field $\phi(x)$. Indeed, we might regard $\langle \phi^2(x)\rangle$ as a sort of poor man's $\langle T_\mu{}^\nu\rangle$ since it provides considerable insight into the physical content of the different vacua. It suffers from the defect, however, of being a scalar, and therefore does not distinguish between the future and past horizons, or between $\mathscr{I}^+$ and $\mathscr{I}^-$.

In the exterior region of Schwarzschild spacetime a complete set of normalized basis functions for the massless scalar field is[10]

$$\vec{u}_{\omega lm}(x)=(4\pi\omega)^{-1/2}e^{-i\omega t}\,\vec{R}_l(\omega\,|\,r)Y_{lm}(\theta,\phi)\,,$$
$$\overleftarrow{u}_{\omega lm}(x)=(4\pi\omega)^{-1/2}e^{-i\omega t}\,\overleftarrow{R}_l(\omega\,|\,r)Y_{lm}(\theta,\phi)\,,$$

which have the asymptotic forms

$$\vec{R}_l(\omega\,|\,r)\sim\begin{cases} r^{-1}e^{i\omega r_*}+\vec{A}_l(\omega)r^{-1}e^{-i\omega r_*}\,, & r\to 2M\\ B_l(\omega)r^{-1}e^{i\omega r_*}\,, & r\to\infty\end{cases}$$

$$\overleftarrow{R}_l(\omega\,|\,r)\sim\begin{cases} B_l(\omega)r^{-1}e^{-i\omega r_*}\,, & r\to 2M\\ r^{-1}e^{-i\omega r_*}+\overleftarrow{A}_l(\omega)r^{-1}e^{i\omega r_*}\,, & r\to\infty\end{cases}$$

in which

$$r_*=r+2M\ln\left(\frac{r}{2M}-1\right)$$

is the Regge-Wheeler coordinate. The Feynman propagators corresponding to the three vacua satisfy the equation

$$\Box G(x,x')=-g^{-1/2}\delta(x,x')$$

and are given, for $t>t'$, by[8]

$$\begin{aligned}
G_B(x,x')&=i\sum_{lm}\int_0^\infty\frac{d\omega}{4\pi\omega}\,e^{-i\omega(t-t')}Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi')[\vec{R}_l(\omega\,|\,r)\vec{R}^*_l(\omega\,|\,r')+\overleftarrow{R}_l(\omega\,|\,r)\overleftarrow{R}^*_l(\omega\,|\,r')]\,,\\
G_U(x,x')&=i\sum_{lm}\int_{-\infty}^\infty\frac{d\omega}{4\pi\omega}\,e^{-i\omega(t-t')}Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi')\left[\frac{\vec{R}_l(\omega\,|\,r)\vec{R}^*_l(\omega\,|\,r')}{1-e^{-2\pi\omega/\kappa}}+\theta(\omega)\overleftarrow{R}_l(\omega\,|\,r)\overleftarrow{R}^*_l(\omega\,|\,r')\right]\,,\\
G_H(x,x')&=i\sum_{lm}\int_{-\infty}^\infty\frac{d\omega}{4\pi\omega}\left[e^{-i\omega(t-t')}Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi')\frac{\vec{R}_l(\omega\,|\,r)\vec{R}^*_l(\omega\,|\,r')}{1-e^{-2\pi\omega/\kappa}}\right.\\
&\qquad\left.+\,e^{+i\omega(t-t')}Y^*_{lm}(\theta,\phi)Y_{lm}(\theta',\phi')\frac{\overleftarrow{R}^*_l(\omega\,|\,r)\overleftarrow{R}_l(\omega\,|\,r')}{e^{2\pi\omega/\kappa}-1}\right]\,,
\end{aligned}\tag{5}$$

where $\kappa=(4M)^{-1}$ is the surface gravity of the hole.

The expectation value of ϕ^2 is related to the Feynman propagator in the corresponding state by the formal expression

$$\langle\phi^2(x)\rangle=-i\lim_{x'\to x}G(x,x')\,.\tag{6}$$

In virtue of (5) we obtain an expression of the form

$$\langle \phi^2(x)\rangle = \int_0^\infty d\omega\, \mu(\omega|r)\,.$$

In the Boulware vacuum, for example, we find

$$\mu_B(\omega|r) = \frac{1}{16\pi^2\omega}\sum_{l=0}^\infty (2l+1)[|\vec{R}_l(\omega|r)|^2 + |\overleftarrow{R}_l(\omega|r)|^2]\,.$$

The high-frequency behavior of $\mu(\omega|r)$ may be inferred by noting that (for $t>0$)

$$G(t,r,\theta,\phi;0,r,\theta,\phi) = i\int_0^\infty d\omega\, e^{-i\omega t}\mu(\omega|r)\,.$$

On the other hand, we know on general grounds that

$$G(t,r,\theta,\phi;0,r,\theta,\phi) \sim \frac{-i}{4\pi^2 t^2(1-2M/r)} \quad \text{as } t\to 0^+\,,$$

and in virtue of the elementary identity

$$\int_0^\infty d\omega\, \omega\, e^{-i\omega t} = -\frac{1}{t^2}$$

it follows that

$$\mu(\omega|r) \sim \frac{\omega}{4\pi^2(1-2M/r)} \quad \text{as } \omega\to\infty\,.$$

We shall regularize (6) by geodesic point separation and take $x=(t,r,\theta,\phi)$ and $x'=(t+\epsilon,r,\theta,\phi)$. The renormalized value of $\langle\phi^2\rangle$ is then given by

$$\langle\phi^2(x)\rangle_{\text{ren}} = \lim_{\epsilon\to 0}\left[\int_0^\infty d\omega\, e^{-i\omega\epsilon}\mu(\omega|r) - \frac{1}{8\pi^2\sigma}\right],$$

where σ denotes the geodetic interval between x and x'. A straightforward calculation reveals that to the required order

$$\frac{1}{2\sigma} = \frac{1}{(1-2M/r)\epsilon^2} + \frac{M^2}{12r^4(1-2M/r)}\,. \tag{7}$$

We find

$$\langle\phi^2(x)\rangle_{\text{ren}} = \int_0^\infty d\omega\left[\mu(\omega|r) - \frac{\omega}{4\pi^2(1-2M/r)}\right] - \frac{M^2}{48\pi^2 r^4(1-2M/r)}\,.$$

Writing this expression out explicitly for each of the three vacua we have

$$\langle B|\phi^2(x)|B\rangle_{\text{ren}} = \frac{1}{16\pi^2}\int_0^\infty\frac{d\omega}{\omega}\left[\sum_{l=0}^\infty(2l+1)[|\vec{R}_l(\omega|r)|^2 + |\overleftarrow{R}_l(\omega|r)|^2] - \frac{4\omega^2}{1-2M/r}\right] - \frac{M^2}{48\pi^2r^4(1-2M/r)}\,,$$

$$\langle U|\phi^2(x)|U\rangle_{\text{ren}} = \frac{1}{16\pi^2}\int_0^\infty\frac{d\omega}{\omega}\left[\sum_{l=0}^\infty(2l+1)\left(\coth\frac{\pi\omega}{\kappa}|\vec{R}_l(\omega|r)|^2 + |\overleftarrow{R}_l(\omega|r)|^2\right) - \frac{4\omega^2}{1-2M/r}\right] - \frac{M^2}{48\pi^2r^4(1-2M/r)}\,, \tag{8}$$

$$\langle H|\phi^2(x)|H\rangle_{\text{ren}} = \frac{1}{16\pi^2}\int_0^\infty\frac{d\omega}{\omega}\left[\coth\frac{\pi\omega}{\kappa}\sum_{l=0}^\infty(2l+1)[|\vec{R}_l(\omega|r)|^2 + |\overleftarrow{R}_l(\omega|r)|^2] - \frac{4\omega^2}{1-2M/r}\right] - \frac{M^2}{48\pi^2r^4(1-2M/r)}\,.$$

Of particular interest are the two asymptotic regimes $r\to 2M$, $r\to\infty$. In order to study these we make use of asymptotic forms that are established in Appendix A:

$$\sum_{l=0}^\infty(2l+1)|\vec{R}_l(\omega|r)|^2 \sim \begin{cases}\dfrac{4\omega^2}{1-2M/r}\,, & r\to 2M\\[2mm] \dfrac{1}{r^2}\displaystyle\sum_{l=0}^\infty(2l+1)|B_l(\omega)|^2\,, & r\to\infty\,,\end{cases}$$

$$\sum_{l=0}^{\infty}(2l+1)|\vec{\mathrm{R}}_l(\omega|r)|^2\sim\begin{cases}\dfrac{1}{4M^2}\sum_{l=0}^{\infty}(2l+1)|B_l(\omega)|^2\,, & r\to 2M\\ 4\omega^2\,, & r\to\infty\,.\end{cases}$$

Note the curious nature of the cancellations that occur as $r\to 2M$. For the Boulware vacuum the leading behavior of the sum over outgoing radial functions cancels against the last term in the integrand. The leading behavior of $\langle B|\phi^2(x)|B\rangle_{\rm ren}$ is therefore determined by the "finite" term

$$\frac{M^2}{48\pi^2 r^4(1-2M/r)}$$

which has its origin in (7). In the other two vacua the leading behavior of the outgoing modes can no longer be canceled, due to the presence of the factor $\coth(\pi\omega/\kappa)$. The frequency integrals in ($8U$) and ($8H$) therefore develop a leading term

$$\frac{1}{2\pi^2(1-2M/r)}\int_0^\infty\frac{d\omega\,\omega}{e^{2\pi\omega/\kappa}-1}=\frac{\kappa^2}{48\pi^2(1-2M/r)}$$

which cancels the effect of the "finite" term.

This suggests that $\langle H|\phi^2(x)|H\rangle_{\rm ren}$ and $\langle U|\phi^2(x)|U\rangle_{\rm ren}$ are finite in the limit $r\to 2M$. To show that this is in fact the case we shall take advantage of the analytic properties of the Schwarzschild manifold in order to derive an expression for the Hartle-Hawking propagator which will prove to be rather more amenable to our needs.

Consider the result of setting $t=-i\zeta$ (ζ real) in the Schwarzschild line element (1) which becomes

$$ds^2=\left(1-\frac{2M}{r}\right)d\zeta^2+\frac{dr^2}{1-2M/r}+r^2(d\theta^2+\sin^2\theta d\phi^2)\,.$$

It has been shown[4,11] that the point $r=2M$ is a regular (as opposed to conical) point of the resulting manifold if the coordinate ζ is made periodic with period $2\pi\kappa^{-1}$.

The Hartle-Hawking propagator satisfies the equation

$$\Box G_H(-i\zeta,r,\theta,\phi;-i\zeta',r',\theta',\phi')=-ir^{-2}\delta(\zeta-\zeta')\delta(r-r')\delta(\Omega,\Omega'),\quad(9)$$

where $\delta(\Omega,\Omega')$ denotes the δ function on the two-sphere and may be expanded in terms of Legendre functions

$$\delta(\Omega,\Omega')=\frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)\qquad(10)$$

with

$$\cos\gamma=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')\,.$$

We may incorporate the desired periodicity with respect to ζ by setting

$$\delta(\zeta-\zeta')=\frac{\kappa}{2\pi}\sum_{n=-\infty}^{\infty}e^{in\kappa(\zeta-\zeta')}\,.\qquad(11)$$

Substituting (10) and (11) into (9) we find that we may expand G_H in the form

$$G_H(-i\zeta,r,\theta,\phi;-i\zeta',r',\theta',\phi')=\frac{i\kappa}{2\pi}\sum_{n=-\infty}^{\infty}e^{in\kappa(\zeta-\zeta')}\sum_{l=0}^{\infty}\frac{(2l+1)}{4\pi}P_l(\cos\gamma)\,\chi_{ln}(r,r')\qquad(12)$$

with χ_{ln} the Green's function for the radial equation of frequency $in\kappa$

$$\left(\frac{d}{dr}(r^2-2Mr)\frac{d}{dr}-l(l+1)-\frac{n^2\kappa^2r^4}{r^2-2Mr}\right)\chi_{ln}(r,r')=-\delta(r-r')\qquad(13)$$

subject to the boundary conditions that $\chi_{ln}(r,r')$ be bounded as $r\to 2M$ and tend to zero as $r\to\infty$.

The homogeneous equation corresponding to (13) is

$$\left(\frac{d}{d\eta}(\eta^2-1)\frac{d}{d\eta}-l(l+1)-\frac{n^2(1+\eta)^4}{16(\eta^2-1)}\right)R=0\,,\qquad(14)$$

the appearance of which has been improved by writing

$$\eta=r/M-1\,.$$

For $n=0$, (14) is soluble in terms of the Legendre functions[12]

$$P_l(\eta)\text{ and }Q_l(\eta)\,.$$

When $n\neq 0$ the solutions of (10) are not simply expressed in terms of known functions.

For $n>0$ we denote by $p_l^n(\eta)$ the solution that remains bounded as $\eta\to 1$ and by $q_l^n(\eta)$ the solution that tends to zero as $\eta\to\infty$, we normalize the p_l^n and q_l^n by requiring

$$p_l^n(\eta)\sim(\eta-1)^{n/2},\quad q_l^n(\eta)\sim(\eta-1)^{-n/2}\quad\text{as }\eta\to1\,. \tag{15}$$

With these conventions it is easily seen that

$$\chi_{ln}=\begin{cases}\dfrac{1}{M}P_l(\eta_<)Q_l(\eta_>) & \text{for } n=0\\[2mm] \dfrac{1}{2|n|M}p_l^{|n|}(\eta_<)q_l^{|n|}(\eta_>) & \text{for } n\neq0\,.\end{cases}$$

Thus (12) becomes

$$G_H(-i\zeta,r,\theta,\phi;-i\zeta',r',\theta',\phi')=\frac{i}{32\pi^2M^2}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)P_l(\eta_<)Q_l(\eta_>)+\frac{i}{32\pi^2M^2}\sum_{n=1}^{\infty}\frac{1}{n}\cos n\kappa(\zeta-\zeta')\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)p_l^n(\eta_<)q_l^n(\eta_>)\,. \tag{16}$$

We have derived the representation (16) in detail since it seems that it may be of some interest in its own right.

For our immediate needs, however, it suffices to have an expression for $G_H(x,x')$ for the case that one of the points, say x', is located on the bifurcation two-sphere of the horizon (Fig. 1). In virtue of (15) we then find that all the terms in (16) with $n\geq1$ are zero. Thus, since $P_l(1)=1$, we have

$$G_H(x,x')=\frac{i}{32\pi^2M^2}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)Q_l(\eta)\,.$$

Further simplification results on recalling Heine's formula[12]

$$\sum_{l=0}^{\infty}(2l+1)P_l(\mu)Q_l(\eta)=\frac{1}{\eta-\mu}\,.$$

Thus we arrive finally at the remarkably simple expression

$$G_H(x,x')=\frac{i}{32\pi^2M(r-M-M\cos\gamma)}\,. \tag{17}$$

We note in passing that although this expression has been derived for the region $r>2M$ it is valid for $r<2M$ by analytic continuation. In particular, we see that (17) is finite at $r=2M$ and $r=0$ (except when the points x and x' are connected by a real null geodesic) even though each of the radial modes $Q_l(\eta)$ is logarithmically infinite at these points.

Separating our points this time in the radial direction we have when x' is a point on the bifurcation two-sphere

$$\langle H|\phi^2(x')|H\rangle_{\text{ren}}=\lim_{r\to2M}\left(\frac{1}{32\pi^2M(r-2M)}-\frac{1}{8\pi^2\sigma(x,x')}\right),$$

a limit which is easily computed yielding

$$\langle H|\phi^2(x')|H\rangle_{\text{ren}}=\frac{1}{192\pi^2M^2}\,.$$

We have thus determined $\langle H|\phi^2(x)|H\rangle_{\text{ren}}$ for points on the bifurcation two-sphere. This result extends to the rest of the horizon in virtue of the invariance of $\langle H|\phi^2(x)|H\rangle_{\text{ren}}$ under the isometries generated by the Killing vector $\partial/\partial t$. We have

$$\frac{\partial}{\partial t}\langle H|\phi^2(x)|H\rangle_{\text{ren}}=0\,.$$

Noting that $\partial/\partial t$ may be written in terms of the Kruskal null coordinates (4) as

$$\frac{\partial}{\partial t}=\kappa\left(V\frac{\partial}{\partial V}-U\frac{\partial}{\partial U}\right),$$

we see immediately that

$$\frac{\partial}{\partial V}\langle H|\phi^2(x)|H\rangle_{\text{ren}}=0\quad\text{when }U=0$$

and

$$\frac{\partial}{\partial U}\langle H|\phi^2(x)|H\rangle_{\text{ren}}=0\quad\text{when }V=0\,.$$

We display the asymptotic forms for $\langle\phi^2\rangle_{\text{ren}}$ in Table I for each of the three vacua. The inter-

TABLE I. The asymptotic values of $\langle\phi^2(x)\rangle_{\text{ren}}$.

	Boulware vacuum	Unruh vacuum	Hartle-Hawking vacuum
$r\to2M$	$-\dfrac{1}{2\pi^2(1-2M/r)}\displaystyle\int_0^\infty\frac{d\omega\,\omega}{e^{2\pi\omega/\kappa}-1}$	$\dfrac{1}{192\pi^2M^2}-\dfrac{1}{32\pi^2M^2}\displaystyle\int_0^\infty\frac{d\omega\sum(2l+1)\lvert B_l(\omega)\rvert^2}{\omega(e^{2\pi\omega/\kappa}-1)}$	$\dfrac{1}{192\pi^2M^2}$
$r\to\infty$	0	$\dfrac{1}{8\pi^2r^2}\displaystyle\int_0^\infty\frac{d\omega\sum(2l+1)\lvert B_l(\omega)\rvert^2}{\omega(e^{2\pi\omega/\kappa}-1)}$	$\dfrac{1}{2\pi^2}\displaystyle\int_0^\infty\frac{d\omega\,\omega}{e^{2\pi\omega/\kappa}-1}$

pretation suggested by these results is as follows:

(i) The Boulware vacuum corresponds to our familiar concept of an empty state for large radii, but is pathological at the horizon in the sense that the renormalized expectation values of physically observable quantities are likely to diverge as the horizon is approached.

(ii) In the Unruh vacuum we find a *flux* of black-body radiation for large radii while $\langle \phi^2(x)\rangle_{\rm ren}$ remains bounded as the horizon is approached.

(iii) The Hartle-Hawking vacuum does not correspond to our usual notion of a vacuum. $\langle \phi^2\rangle_{\rm ren}$ is well behaved on the horizon, but the state is not empty at infinity corresponding instead to a thermal distribution of (Minkowski-type) quanta at the black-hole temperature. That is, the Hartle-Hawking vacuum corresponds to a black hole in (unstable) equilibrium with an infinite sea of black-body radiation.

We shall refine these observations when we come to study the asymptotic forms for $\langle T_\mu{}^\nu\rangle_{\rm ren}$. However, even on the basis of our results thus far we would expect that it is the Unruh vacuum that best approximates the state that would obtain following the gravitational collapse of a massive body.

III. THE ASYMPTOTIC VALUES OF $\langle T_\mu{}^\nu\rangle_{\rm ren}$: RESULTS OBTAINABLE WITHOUT RECOURSE TO RENORMALIZATION

The expected regularity of $\langle H|T_\mu{}^\nu|H\rangle_{\rm ren}$ on both the future and past horizons in a freely falling frame implies that its components in Schwarzschild coordinates also have a finite limit as $r\to 2M$. This feature enables us to compute the leading behavior of $\langle B|T_\mu{}^\nu|B\rangle_{\rm ren}$ and the t and r components of $\langle U|T_\mu{}^\nu|U\rangle_{\rm ren}$ near the horizon since these quantities diverge as $r\to 2M$. Thus we have

$$\langle B|T_\mu{}^\nu|B\rangle_{\rm ren} \underset{r\to 2M}{\sim} \langle B|T_\mu{}^\nu|B\rangle_{\rm ren}-\langle H|T_\mu{}^\nu|H\rangle_{\rm ren}=\langle B|T_\mu{}^\nu|B\rangle-\langle H|T_\mu{}^\nu|H\rangle$$

$$\underset{r\to 2M}{\sim} -\frac{1}{2\pi^2(1-2M/r)^2}\int_0^\infty \frac{d\omega\,\omega^3}{e^{2\pi\omega/\kappa}-1}\begin{pmatrix}-1&0&0&0\\0&\frac13&0&0\\0&0&\frac13&0\\0&0&0&\frac13\end{pmatrix}$$

in Schwarzschild coordinates, the last line following from a mode sum that is evaluated in Appendix A. We observe that this stress tensor is infinite on both horizons and corresponds, in some sense, to the absence from the vacuum of black-body radiation at the black-hole temperature. This result bears a very close resemblance to the vacuum stress above an infinitely accelerated barrier.[9,13] It is reasonable to conclude from this that a physical realization of the Boulware vacuum would be the vacuum state outside a massive spherical body of radius only slightly larger than its Schwarzschild radius.

In fact, on the basis of the similarity between the result for $\langle B|T_\mu{}^\nu|B\rangle_{\rm ren}$ as $r\to 2M$ and the form of $\langle T_\mu{}^\nu\rangle_{\rm ren}$ above an accelerated conductor we are led to conjecture that for a massless field of spin s

$$\langle B|T_\mu{}^\nu|B\rangle_{\rm ren} \underset{r\to 2M}{\sim} \frac{-h(s)}{2\pi^2(1-2M/r)^2}\times\int_0^\infty\frac{d\omega\,\omega(\omega^2+s^2\kappa^2)}{e^{2\pi\omega/\kappa}-(-1)^{2s}}\times\begin{pmatrix}-1&0&0&0\\0&\frac13&0&0\\0&0&\frac13&0\\0&0&0&\frac13\end{pmatrix},$$

where $h(s)$ is the number of helicity states for a massless field of spin s. I have verified this conjecture by direct calculation for the case $s=1$.[14]

For the Unruh vacuum we have

$$\langle U|T_a{}^b|U\rangle_{\rm ren} \underset{r\to 2M}{\sim} \langle U|T_a{}^b|U\rangle-\langle H|T_a{}^b|H\rangle \sim \frac{L}{4\pi}\begin{pmatrix}\left(1-\frac{2M}{r}\right)^{-1} & -r^{-2}\\ r^{-2}\left(1-\frac{2M}{r}\right)^{-2} & -\left(1-\frac{2M}{r}\right)^{-1}\end{pmatrix}, \tag{18}$$

where a and b range over t and r, and the luminosity of the hole L is given by

$$L=\frac{1}{2\pi}\int_0^\infty\frac{d\omega\,\omega\sum_{l=0}^\infty(2l+1)|B_l(\omega)|^2}{e^{2\pi\omega/\kappa}-1}.$$

Now the regularity of of $\langle H|T_\theta{}^\theta|H\rangle_{\rm ren}$ on the horizon ensures that of $\langle U|T_\theta{}^\theta|U\rangle_{\rm ren}$ since these quantities differ by a finite amount. From this we observe from (18) that $\langle U|T_\mu{}^\nu|U\rangle_{\rm ren}$ is regular, in a freely falling frame, on the future horizon but diverges on the past horizon.

As $r\to\infty$, $\langle B|T_\mu{}^\nu|B\rangle_{\rm ren}$ is expected to be of the order of the square of the Riemann tensor, i.e., $O(M^2r^{-6})$, thus we may write

$$\langle H|T_\mu{}^\nu|H\rangle_{\rm ren} \underset{r\to\infty}{\sim} \langle H|T_\mu{}^\nu|H\rangle - \langle B|T_\mu{}^\nu|B\rangle$$

$$\to \frac{1}{2\pi^2}\int_0^\infty \frac{d\omega\omega^3}{e^{2\pi\omega/\kappa}-1} \times \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{bmatrix} \qquad (19)$$

and

$$\langle U|T_\mu{}^\nu|U\rangle_{\rm ren} \underset{r\to\infty}{\sim} \langle U|T_\mu{}^\nu|U\rangle - \langle B|T_\mu{}^\nu|B\rangle$$

$$\sim \frac{L}{4\pi r^2}\begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (20)$$

where the final expressions in (19) and (20) again result from explicit evaluations of mode sums that are performed in Appendix A. We note that (19) corresponds to a bath of blackbody radiation at the black-hole temperature while (20) corresponds to a flux of radiation outgoing at $\mathscr{I}^+$. Thus we have confirmed that the asymptotic behavior of $\langle T_\mu{}^\nu\rangle$ is as was conjectured by Christensen and Fulling.[8]

IV. $\langle H|T_\mu{}^\nu|H\rangle_{\rm ren}$ ON THE HORIZON

We turn now to the explicit evaluation of $\langle H|T_\mu{}^\nu|H\rangle_{\rm ren}$ on the horizon. We shall in the first instance calculate $\langle H|T_\mu{}^\nu|H\rangle_{\rm ren}$ on the bifurcation two-sphere basing the calculation on the representation (16). The result may then be partially extended to the rest of the horizon in virtue of the symmetry generated by the Killing vector $\partial/\partial t$. Unfortunately, however, one component of $\langle H|T_\mu{}^\nu|H\rangle_{\rm ren}$ remains undetermined by this process. I hope to return to the computation of this remaining component in a future publication.

As has been remarked previously, the evaluation of $\langle H|T_\mu{}^\nu|H\rangle_{\rm ren}$ on the bifurcation two-sphere will require us to differentiate the representation (16) before allowing η' to approach unity. One result of this is that we shall be obliged to evaluate angular sums over the radial functions with $n=1$, a task somewhat more involved than that of evaluating the corresponding sums for $n=0$.

We shall again regularize by separating points along a radial geodesic. For brevity let us introduce the notation

$$[G], [G_{;\mu}], [G_{;\mu\nu'}], \text{ etc.}$$

to denote the indicated two-point functions evaluated in the *partial coincidence limit*

$$\zeta=\zeta', \quad \theta=\theta', \quad \phi=\phi'.$$

With this understanding we write the point-separated stress tensor in the form

$$\begin{aligned}\langle T_{\mu'\nu'}\rangle_{\rm separated} = -i\{&\tfrac{1}{3}([G_{;\mu'\alpha}g^\alpha{}_{\nu'} + G_{;\nu'\alpha}g^\alpha{}_{\mu'}]) \\ &- \tfrac{1}{6}[G_{;\alpha\beta'}g^{\alpha\beta'}g_{\mu'\nu'}] \\ &- \tfrac{1}{6}([G_{;\mu'\nu'}] \\ &+ [G_{;\alpha\beta}g^\alpha{}_{\mu'}g^\beta{}_{\nu'}])\}, \end{aligned} \qquad (21)$$

where $g_\alpha{}^{\beta'}$ is the bivector of parallel transport.

It is convenient to introduce a modified Kruskal "time" coordinate y by writing

$$v=-iy.$$

Recalling that $t=-i\zeta$, we may replace (2) by

$$\begin{aligned} y &= \left(\frac{r}{2M}-1\right)^{1/2} e^{\kappa r}\sin(\kappa\zeta), \\ u &= \left(\frac{r}{2M}-1\right)^{1/2} e^{\kappa r}\cos(\kappa\zeta). \end{aligned} \qquad (22)$$

The obvious analogy between (22) and the relation between Cartesian and plane polar coordinates prompts us to define yet another radial variable ρ by

$$\rho = \left(\frac{r}{2M}-1\right)^{1/2} e^{\kappa r} \qquad (23)$$

in terms of which the metric becomes

$$\begin{aligned} ds^2 = &\frac{32M^3}{r}e^{-2\kappa r}(d\rho^2+\rho^2\kappa^2 d\zeta^2) \\ &+ r^2(d\theta^2+\sin^2\theta d\phi^2). \end{aligned}$$

It is also convenient to adopt a suffix convention whereby

a,b,c,d range over (u,y),

i,j,k,l range over (θ,ϕ),

and Greek suffices range over all four coordinates as usual.

The symmetries are such that on the bifurcation two-sphere $\langle H|T_{\mu'\nu'}|H\rangle_{\rm ren}$ must assume the form

$$\langle H|T_{\mu'\nu'}|H\rangle_{\rm ren} = \begin{bmatrix} Ag_{a'b'} & 0 \\ 0 & Bg_{j'k'} \end{bmatrix}, \qquad (24)$$

where A and B are constants whose sum is determined by the trace anomaly

$$\begin{aligned}\langle H|T_{\mu'}{}^{\mu'}|H\rangle_{\rm ren} &= \frac{1}{2880\pi^2}C^{\alpha'\beta'\gamma'\delta'}C_{\alpha'\beta'\gamma'\delta'} \\ &= \frac{1}{3840\pi^2 M^4}\end{aligned}$$

when $r' = 2M$.

The determination of the constants A and B in (24) devolves upon the computation and renormalization of the point-separated stress tensor (21). The calculation involved is straightforward in principle but since it is somewhat beset by detail we shall pause here to indicate our strategy.

We shall first calculate the bivectors of parallel transport for two points

$$x = (\zeta, r, \theta, \phi), \quad x' = (\zeta, r', \theta, \phi)$$

which have the same values for ζ, θ, ϕ and where $r > 2M$ and $r' > 2M$. A considerable virtue of point separating in the radial direction is that the $g_\mu{}^{\nu'}$ assume a very simple form.

The next step is to calculate the various combinations of second derivatives of G that we will require. Some of these may be obtained by direct differentiation of (17) while others involve the radial functions corresponding to $n=1$. In particular, it is necessary to obtain an asymptotic expansion of the quantity

$$\sum_{l=0}^{\infty} (2l+1) q_l^1(\eta)$$

as $\eta \to 1$. This in turn requires the development of uniform asymptotic expansions for the radial functions $q_l^1(\eta)$. Marshalling these various quantities we are able to calculate the point-separated stress tensor (21). It will then remain only to evaluate the subtraction terms $\langle T_{\mu\nu}\rangle_{\text{subtract}}$ (Ref. 6) and to calculate the limit

$$\langle H | T_{a'}{}^{a'} | H \rangle_{\text{ren}} = \lim_{r \to 2M} (\langle H | T_{a'}{}^{a'} | H \rangle_{\text{separated}} - \langle T_{a'}{}^{a'} \rangle_{\text{subtract}}). \tag{25}$$

Determination of the $g_\mu{}^{\nu'}$

In virtue of the symmetries of the manifold we must have

$$\frac{D}{\partial\rho}\frac{\partial}{\partial\rho} = \alpha(\rho)\frac{\partial}{\partial\rho} \tag{26}$$

for some function $\alpha(\rho)$.

Furthermore, since $g_{\rho\zeta} = 0$ we have

$$\begin{aligned} 0 &= \frac{\partial}{\partial\rho}\underline{g}\left(\frac{\partial}{\partial\rho}, \frac{1}{\rho}\frac{\partial}{\partial\zeta}\right) \\ &= \underline{g}\left(\frac{D}{\partial\rho}\frac{\partial}{\partial\rho}, \frac{1}{\rho}\frac{\partial}{\partial\zeta}\right) + \underline{g}\left(\frac{\partial}{\partial\rho}, \frac{D}{\partial\rho}\ \frac{1}{\rho}\frac{\partial}{\partial\zeta}\right) \\ &= \underline{g}\left(\frac{\partial}{\partial\rho}, \frac{D}{\partial\rho}\ \frac{1}{\rho}\frac{\partial}{\partial\zeta}\right) \end{aligned}$$

from which, again invoking the symmetries of the manifold, we may deduce that

$$\frac{D}{\partial\rho}\left(\frac{1}{\rho\kappa}\frac{\partial}{\partial\zeta}\right) = \beta(\rho)\frac{1}{\rho\kappa}\frac{\partial}{\partial\zeta} \tag{27}$$

for some function $\beta(\rho)$. Now

$$\begin{aligned} \frac{\partial}{\partial\rho} g_{\rho\rho} &= \frac{\partial}{\partial\rho}\underline{g}\left(\frac{\partial}{\partial\rho}, \frac{\partial}{\partial\rho}\right) \\ &= 2\underline{g}\left(\frac{D}{\partial\rho}\frac{\partial}{\partial\rho}, \frac{\partial}{\partial\rho}\right) \\ &= 2\alpha(\rho) g_{\rho\rho} \end{aligned} \tag{28}$$

and

$$\begin{aligned} \frac{\partial}{\partial\rho}\frac{1}{\rho^2\kappa^2} g_{\zeta\zeta} &= \frac{\partial}{\partial\rho}\underline{g}\left(\frac{1}{\rho\kappa}\frac{\partial}{\partial\zeta}, \frac{1}{\rho\kappa}\frac{\partial}{\partial\zeta}\right) \\ &= 2\underline{g}\left(\frac{D}{\partial\rho}\left(\frac{1}{\rho\kappa}\frac{\partial}{\partial\zeta}\right), \frac{1}{\rho\kappa}\frac{\partial}{\partial\zeta}\right) \\ &= 2\beta(\rho)\frac{1}{\rho^2\kappa^2} g_{\zeta\zeta}, \end{aligned} \tag{29}$$

but since we also have

$$g_{\zeta\zeta} = \rho^2\kappa^2 g_{\rho\rho}$$

it follows from (28) and (29) that

$$\alpha(\rho) = \beta(\rho) = \frac{1}{2}\frac{\partial}{\partial\rho}\ln g_{\rho\rho}. \tag{30}$$

In order to apply these results in a more standard coordinate system we note that in virtue of (22) and (23) we have

$$\begin{aligned} \frac{\partial}{\partial u} &= \cos(\kappa\zeta)\frac{\partial}{\partial\rho} - \frac{\sin(\kappa\zeta)}{\rho\kappa}\frac{\partial}{\partial\zeta}, \\ \frac{\partial}{\partial y} &= \sin(\kappa\zeta)\frac{\partial}{\partial\rho} + \frac{\cos(\kappa\zeta)}{\rho\kappa}\frac{\partial}{\partial\zeta} \end{aligned} \tag{31}$$

from which it is clear that

$$\frac{D}{\partial\rho}\frac{\partial}{\partial u} = \alpha(\rho)\frac{\partial}{\partial u} \quad\text{and}\quad \frac{D}{\partial\rho}\frac{\partial}{\partial y} = \alpha(\rho)\frac{\partial}{\partial y},$$

and given (30) it follows rapidly that in Kruskal coordinates

$$g_a{}^{b'} = \begin{cases} (g_{aa} g^{b'b'})^{1/2} & \text{if } a = b \\ 0 & \text{if } a \neq b. \end{cases}$$

Similar reasoning reveals that

$$g_j{}^{k'} = \begin{cases} (g_{jj} g^{k'k'})^{1/2} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

and that the other components of $g_\mu{}^{\nu'}$ vanish.

The derivatives of $G(x,x')$

For completeness we record here the nonvanishing components of the connection:

$$\begin{aligned} &\Gamma_{tr}{}^{t} = \frac{M}{r^2(1-2M/r)}, \\ &\Gamma_{tt}{}^{r} = \frac{M}{r^2}\left(1 - \frac{2M}{r}\right), \quad \Gamma_{rr}{}^{r} = -\frac{M}{r^2(1-2M/r)}, \\ &\Gamma_{\theta\theta}{}^{r} = -r\left(1 - \frac{2M}{r}\right), \quad \Gamma_{\phi\phi}{}^{r} = -r\sin^2\theta\left(1 - \frac{2M}{r}\right), \\ &\Gamma_{r\theta}{}^{\theta} = \frac{1}{r}, \quad \Gamma_{\phi\phi}{}^{\theta} = -\sin\theta\cos\theta, \\ &\Gamma_{r\phi}{}^{\phi} = \frac{1}{r}, \quad \Gamma_{\theta\phi}{}^{\phi} = \cot\theta. \end{aligned} \tag{32}$$

Let us first consider quantities of the form

$$[G_{;jk}],\quad [G_{;jk'}],\quad [G_{;j'k'}].$$

It is evident that the first two of these may be obtained by direct differentiation of (17). The $G_{;j'k'}$ potentially involve the $G_{,a'}$ since there are nonvanishing connection coefficients of the form $\Gamma_{j'k'}{}^{a'}$. However, we have

$$G_{;j'k'}=G_{,j'k'}-\Gamma_{j'k'}{}^{\mu'}G_{,\mu'},$$

and using (32) we may rewrite the second term as

$$\Gamma_{j'k'}{}^{\mu'}G_{,\mu'}=-g_{j'k'}\frac{1}{r'}\left(1-\frac{2M}{r'}\right)\frac{\partial G}{\partial r'}+\Gamma_{j'k'}{}^{i'}\,G_{,i'}.$$

Now we see from the representation (16) that

$$\left(1-\frac{2M}{r'}\right)\frac{\partial G}{\partial r'}\to 0 \text{ as } r'\to 2M,$$

and since we also have

$$[G_{,i'}]=0$$

we may deduce that

$$[G_{;j'k'}]=[G_{,j'k'}].$$

Thus the $G_{;j'k'}$ may also be obtained by direct differentiation of (17).

We now turn to the quantities

$$[G_{;ab}],\quad [G_{;ab'}],\quad [G_{;a'b'}].$$

Of these the first may again be obtained by direct differentiation of (17) while the second and third are more complicated. However, since it proves sufficient for our purposes to have a knowledge of the (two-dimensional) traces of these quantities we may evaluate the relevant part of the $[G_{;a'b'}]$ by invoking the wave equation. Thus we have

$$g^{a'b'}[G_{;a'b'}]=-g^{j'k'}[G_{;j'k'}],$$

a quantity which we have already evaluated.

Thus of the derivatives of G it remains now only to calculate the $[G_{;ab'}]$. Referring back to (16) we see that this quantity devolves upon a determination of the limits

$$\lim_{\eta'\to 1}\nabla_{b'}P_l(\eta') \tag{33}$$

and

$$\lim_{\eta'\to 1}\nabla_{b'}p_l^n(\eta')\cos\eta\kappa(\zeta-\zeta') \tag{34}$$

in which we shall take b to denote either of the Kruskal coordinates. We note that as $\eta'\to 1$

$$P_l(\eta')=1+O(\rho'^2)$$

and

$$p_l^n(\eta')\sim(2^{1/2}e^{-1/2}\rho')^n.$$

Therefore, by (31), we infer that (33) is zero and that (34) is nonvanishing only for $n=1$, and that for this case we have

$$\lim_{\eta'\to 1}\frac{\partial}{\partial u'}p_l{}^1(\eta')\cos(\zeta-\zeta')=2^{1/2}e^{-1/2}\cos\kappa\zeta$$

and

$$\lim_{\eta'\to 1}\frac{\partial}{\partial y'}p_l{}^1(\eta')\cos\kappa(\zeta-\zeta')=2^{1/2}e^{-1/2}\sin\kappa\zeta.$$

From which it rapidly follows that

$$-32i\pi^2M^2[G_{;a'c}g^c{}_{b'}]=\frac{e^{\kappa r-1}}{2}\left(\frac{r}{M}\right)^{1/2}\left(\frac{\partial F}{\partial\rho}+\frac{F}{\rho}\right)\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{e^{\kappa r-1}}{2}\left(\frac{r}{M}\right)^{1/2}\left(\frac{\partial F}{\partial\rho}-\frac{F}{\rho}\right)\times\begin{bmatrix}\cos2\kappa\zeta&\sin2\kappa\zeta\\ \sin2\kappa\zeta&-\cos2\kappa\zeta\end{bmatrix},$$

where

$$F(\eta)=\sum_{l=0}^{\infty}(2l+1)q_l^1(\eta). \tag{35}$$

The results for the various coincidence limits of the derivatives of G have been gathered together in Table II.

Rather than calculate all the components of $\langle H|T_{\mu'\nu'}|H\rangle_{\text{separated}}$ directly it proves sufficient for our needs, and somewhat simpler, to calculate the partial trace $\langle H|T_{a'}{}^{a'}|H\rangle_{\text{separated}}$. From (21) this is given by

TABLE II. Partial coincidence limits of the two-dimensional traces of the second derivatives of $G_H(x,x')$ for x' a point on the bifurcation two-sphere.

$\vec{\Theta}$	$g^{ab}\nabla_{ab}$	$g^{ab'}\nabla_{ab'}$	$g^{a'b'}\nabla_{a'b'}$
$-32i\pi^2M^2[\vec{\Theta}G]$	$\dfrac{2M(r-M)}{r^2(r-2M)^2}$	$\dfrac{e^{\kappa r}}{16M^2}\left(\dfrac{r}{M}\right)^{1/2}\left[\dfrac{\partial F}{\partial\rho}+\dfrac{F}{\rho}\right]$	$\dfrac{1}{2(r-2M)^2}$
$\vec{\Theta}$	$g^{jk}\nabla_{jk}$	$g^{jk'}\nabla_{jk'}$	$g^{j'k'}\nabla_{j'k'}$
$-32i\pi^2M^2[\vec{\Theta}G]$	$\dfrac{-2M(r-2M)}{r^2(r-2M)^2}$	$\dfrac{M}{r(r-2M)^2}$	$-\dfrac{1}{2(r-2M)^2}$

$$\langle H|\, T_{a'}{}^{a'}\, |H\rangle_{\text{separated}} = -i\{\tfrac{1}{3}([G_{;a'b}\, g^{a'b}] - [G_{;j'k}\, g^{j'k}]) - \tfrac{1}{6}([G_{;a'}{}^{a'}] + [G_{;a}{}^{a}])\}. \tag{36}$$

Substituting for the derivatives of G we find

$$\langle H|\, T_{a'}{}^{a'}\, |H\rangle_{\text{separated}} = \frac{e^{\kappa r}}{1536\pi^2 M^4}\left(\frac{r}{M}\right)^{1/2}\left(\frac{\partial F}{\partial\rho} + \frac{F}{\rho}\right) - \frac{r^2 + 8Mr - 4M^2}{384\pi^2 M^2 r^2 (r-2M)^2}. \tag{37}$$

It is now incumbent upon us to evaluate $\langle T_{\mu'\nu'}\rangle_{\text{subtract}}$ the subtraction terms of Christensen and subtract these from (37) in order to evaluate $\langle H|\, T_{a'}{}^{a'}\, |H\rangle_{\text{ren}}$. From Ref. 6 we find that for a Ricci flat space

$$\begin{aligned}
-8\pi^2\langle T_{\mu'\nu'}\rangle_{\text{subtract}} &= \frac{-4}{\sigma_{\rho'}\sigma^{\rho'}}\left(g_{\mu'\nu'} - 4\frac{\sigma_{\mu'}\sigma_{\nu'}}{\sigma_{\rho'}\sigma^{\rho'}}\right) + \tfrac{1}{360} R^{\rho'\tau'\kappa'\iota'} R_{\rho'\tau'\kappa'\iota'} \frac{\sigma_{\mu'}\sigma_{\nu'}}{\sigma_{\rho'}\sigma^{\rho'}} \\
&\quad + \left(\tfrac{1}{45} R_{\mu'}{}^{\rho'}{}_{\nu'}{}^{\tau'} R_{\rho'\alpha'\tau'\beta'} + \tfrac{1}{45} R_{\mu'\tau'\rho'\alpha'} R_{\nu'}{}^{\rho'\tau'}{}_{\beta'} + \tfrac{1}{45} R_{\mu'\rho'\tau'\alpha'} R_{\nu'}{}^{\rho'\tau'}{}_{\beta'}\right) \frac{\sigma^{\alpha'}\sigma^{\beta'}}{\sigma_{\rho'}\sigma^{\rho'}} \\
&\quad - \tfrac{1}{45} R^{\rho'\tau'\kappa'}{}_{\alpha'} R_{\rho'\tau'\kappa'(\mu'} \frac{\sigma_{\nu')}\sigma^{\alpha'}}{\sigma_{\rho'}\sigma^{\rho'}} - \tfrac{1}{180} R_{\rho'\tau'\kappa'\alpha'} R^{\rho'\tau'\kappa'}{}_{\beta'} \frac{\sigma^{\alpha'}\sigma^{\beta'}}{\sigma_{\rho'}\sigma^{\rho'}}\left(g_{\mu'\nu'} - 2\frac{\sigma_{\mu'}\sigma_{\nu'}}{\sigma_{\rho'}\sigma^{\rho'}}\right) \\
&\quad - \tfrac{4}{45} R_{\rho'\beta'\tau'\gamma'} R^{\tau'}{}_{\alpha'}{}^{\rho'}{}_{(\mu'} \frac{\sigma_{\nu')}\sigma^{\alpha'}\sigma^{\beta'}\sigma^{\gamma'}}{(\sigma_{\rho'}\sigma^{\rho'})^2} - \tfrac{1}{90} R^{\rho'}{}_{\alpha'}{}^{\tau'}{}_{\beta'} R_{\rho'\gamma'\tau'\delta'} \frac{\sigma^{\alpha'}\sigma^{\beta'}\sigma^{\gamma'}\sigma^{\delta'}}{(\sigma_{\rho'}\sigma^{\rho'})^2}\left(g_{\mu'\nu'} - 4\frac{\sigma_{\mu'}\sigma_{\nu'}}{\sigma_{\rho'}\sigma^{\rho'}}\right) \\
&\quad + \left(\tfrac{1}{90} R_{\mu'\alpha'\nu'\beta';\gamma'\delta'} - \tfrac{1}{45} R^{\rho'}{}_{\alpha'\mu'\beta'} R_{\rho'\gamma'\nu'\delta'} + \tfrac{1}{36} R^{\rho'}{}_{\alpha'}{}^{\tau'}{}_{\beta'} R_{\rho'\gamma'\tau'\delta'} g_{\mu'\nu'}\right) \frac{\sigma^{\alpha'}\sigma^{\beta'}\sigma^{\gamma'}\sigma^{\delta'}}{(\sigma_{\rho'}\sigma^{\rho'})^2}.
\end{aligned} \tag{38}$$

The evaluation of the above is a straightforward calculation which is simplified by the observation that the nonvanishing components of the Riemann tensor may be expressed in the form

$$\begin{aligned}
R_{abcd} &= \frac{16M^4}{r^6}(g_{ac}g_{bd} - g_{ad}g_{bc}), \\
R_{ajbk} &= \frac{8M^4}{r^6} g_{ab} g_{jk}, \\
R_{ijkl} &= \frac{16M^4}{r^6}(g_{ik}g_{jl} - g_{il}g_{jk}).
\end{aligned} \tag{39}$$

For completeness we compute the values of the individual terms in Appendix B. The result of this calculation is that the first term in (38) which is the quartic infinity gives rise to a term

$$\langle T_{a'}{}^{a'}\rangle_{\text{quartic subtract}} = -\frac{1}{64\pi^2 M^2 (r-2M)^2} + \frac{1}{192\pi^2 M^3 (r-2M)} - \frac{17}{11520\pi^2 M^4} + O(r-2M). \tag{40}$$

The finite direction-dependent terms in (38) give rise to a contribution

$$\langle T_{a'}{}^{a'}\rangle_{\text{finite subtract}} = -\frac{1}{2880\pi^2 M^4}. \tag{41}$$

Thus subtracting (40) and (41) from (37) we arrive at the expression

$$\langle H|\, T_{a'}{}^{a'}\, |H\rangle_{\text{ren}} = \frac{1}{768\pi^2 M^4} \times \lim_{r\to 2M}\left[\frac{e^{\kappa r}}{2}\left(\frac{r}{M}\right)^{1/2}\left(\frac{\partial F}{\partial\rho} + \frac{F}{\rho}\right) + \frac{4M^2}{(r-2M)^2} - \frac{2M}{(r-2M)} + \frac{9}{10}\right]. \tag{42}$$

It remains now only to develop an asymptotic expansion for F as far as the order that is required to perform the limit (42), i.e., $O((r-2M)^{1/2})$. To this end we shall develop asymptotic expansions of the radial functions $q_l^1(\eta)$ that are uniformly valid with respect to the angular momentum parameter l.

Consider the radial equation (14) for values of η very close to unity. In particular, we observe that the last term in this equation contains the factor

$$\frac{(1+\eta)^4}{16(\eta^2-1)} \underset{\eta\to 1}{\sim} \frac{1}{\eta^2-1}.$$

Thus there is some sense in which $q_l^n(\eta)$ is asymptotic, as $\eta\to 1$, to a solution of

$$\left(\frac{d}{d\eta}(\eta^2-1)\frac{d}{d\eta} - l(l+1) - \frac{n^2}{\eta^2-1}\right)u = 0 \tag{43}$$

which we recognize as the associated Legendre equation corresponding to parameters (l, n). Therefore, recalling that

$$q_l^1(\eta) \sim (\eta-1)^{-1/2}$$

as $\eta \to 1$ *for fixed* l we see that there is a sense in which a uniformly valid asymptotic form for q_l^1 is given by

$$q_l^1(\eta) \underset{\eta\to 1}{\sim} -\sqrt{2}\, Q_l^1(\eta)\,. \tag{44}$$

The freedom to add a multiple of the "small" solution $P_l^1(\eta)$ of (43) to the right-hand side of (44) is a point that will be subsumed in what follows.

It turns out, perhaps somewhat surprisingly, that although we have to go to an asymptotic approximation of higher order than (44) in order to correctly compute the limit (42), the lowest-order asymptotic form (44) already suffices to correctly determine the infinite part of $\langle T_{\mu'\nu'}\rangle$.

In order to improve on (44) we may, with a certain prescience, write

$$q_l^1(\eta) = \beta_l\, p_l^1(\eta) - \sqrt{2}\, Q_l^1(\eta) - f(\eta) Q_l(\eta) + \tilde{q}_l^1(\eta), \tag{45}$$

where the β_l are constants and $f(\eta)$ is a function independent of l. We shall now show that it is possible to choose $f(\eta)$ and the coefficients β_l in such a way that the "remainder" terms $\tilde{q}_l^1(\eta)$ do not contribute to the limit (42).

Let us denote the differential operator (14) specialized to the case $n=1$ by $\vec{\mathrm{L}}$. Then in virtue of the properties of the Legendre functions and the fact that $\vec{\mathrm{L}} q_l^1$ and $\vec{\mathrm{L}} p_l^1$ both vanish we find that

$$\vec{\mathrm{L}}\tilde{q}_l^1(\eta) = \frac{Q_l^1(\eta)\sqrt{2}\,[16-(1+\eta)^4]}{16(\eta^2-1)} + Q_l(\eta)\left(\frac{d}{d\eta}(\eta^2-1)\frac{d}{d\eta} - \frac{(1+\eta)^4}{16(\eta^2-1)}\right) f(\eta) + 2(\eta^2-1)\frac{d}{d\eta} Q_l(\eta)\frac{d}{d\eta} f(\eta)\,. \tag{46}$$

Now

$$\frac{d}{d\eta} Q_l(\eta) = (\eta^2-1)^{-1/2} Q_l^1(\eta)\,. \tag{47}$$

Thus by an appropriate choice of $f(\eta)$ we can arrange for the third term on the right-hand side of (46) to cancel the first. This is accomplished by choosing

$$f(\eta) = \frac{\sqrt{2}}{32}\int_1^\eta \frac{d\xi}{(\xi^2-1)^{3/2}}\,[(1+\xi)^4 - 16]\,. \tag{48}$$

With this choice then

$$\vec{\mathrm{L}}\tilde{q}_l^1(\eta) = Q_l(\eta)\left(\frac{d}{d\eta}(\eta^2-1)\frac{d}{d\eta} - \frac{(1+\eta)^4}{16(\eta^2-1)}\right) f(\eta)\,. \tag{49}$$

From (48) and (49) it is readily shown that

$$\vec{\mathrm{L}}\tilde{q}_l^1(\eta) \underset{\eta\to 1}{\sim} -\tfrac{9}{112}\,(\eta-1)^{5/2} Q_l(\eta)\,, \tag{50}$$

this relation being uniformly valid in l. By considering (50) as $\eta\to 1$ for fixed l we see that there exists a solution to (49) which satisfies the asymtotic condition

$$\tilde{q}_l^1(\eta) \underset{\eta\to 1}{\sim} \tfrac{3}{1792}\,(\eta-1)^{7/2}\ln(\eta-1)\,. \tag{51}$$

We remark (i) that the differential equation (49) subject to the boundary condition (51) possesses a unique solution, and (ii) that no loss of generality is entailed by taking $\tilde{q}_l^1$ to be this solution since, in virtue of (16), (45), and (49), the only freedom that remains in the choice of the $\tilde{q}_l^1$ consists of the possible addition of multiples of the p_l^1. This amounts to no more than a redefinition of the, as yet unspecified, constants β_l. These are now determined by the imposition of the condition that $q_l^1(\eta)$ should vanish as $\eta\to\infty$.

Consideration of (50) for large values of l reveals that (51) provides a tolerable approximation for values of $l \lesssim \Lambda$ with

$$\Lambda = (\eta-1)^{-1/2}\,;$$

for $l \gtrsim \Lambda$ the combination $\tilde{q}_l^1(\eta) + \beta_l p_l^1(\eta)$ cuts off exponentially in l. Therefore, we are able to estimate the contribution of the terms $\tilde{q}_l^1(\eta) + \beta_l p_l^1(\eta)$ to the sum (35) to leading order in $(\eta-1)$:

$$\sum_{l=0}^{\infty}(2l+1)[\tilde{q}_l^1(\eta) + \beta_l p_l^1(\eta)] \sim (\eta-1)^{1/2}\sum_{l=0}^{\infty}(2l+1)\beta_l + O((\eta-1)^{3/2})\,. \tag{52}$$

Thus the $\tilde{q}_l^1(\eta)$ can be said not to contribute to the limit (42). The convergence of the sum over β_l presents no problems since it can be shown that β_l is of order l^{-6} for large l.

Putting these various results together, recalling Heine's formula from Sec. II, and using (47) we find

$$F(\eta) = \frac{2}{(\eta-1)^{3/2}} - \frac{1}{2(\eta-1)^{1/2}} + (\eta-1)^{1/2}\left[\sum_{l=0}^{\infty}(2l+1)\beta_l - \tfrac{1}{16}\right] + O((\eta-1)^{3/2}\ln(\eta-1))\,. \tag{53}$$

Thus we are finally in a position to compute the renormalized value of the stress tensor. Substituting (53) into (42) we find the value of the constant A occurring in (24):

$$A = \frac{1}{768\pi^2 M^4}\left(-\tfrac{1}{20} + \sum_{l=0}^{\infty}(2l+1)\beta_l\right).$$

The value of B follows immediately from the known value of the trace anomaly:

$$B=\frac{1}{768\pi^2M^4}\left(\frac{3}{20}-\sum_{l=0}^{\infty}(2l+1)\beta_l\right).$$

In Table III we display the results of a numerical computation of the β_l. The details of this calculation may be found in Appendix C.

From the Table III we find

$$\sum(2l+1)\beta_l \simeq 0.1286.$$

We observe from this that both A and B are positive or equivalently (referring now to the physical manifold) that the energy density is negative and the principal pressures are positive.

The extension of $\langle T_{\mu\nu}\rangle_{\rm ren}$ away from the bifurcation two-sphere

An elementary calculation reveals that if $\langle T_{\mu\nu}(x)\rangle_{\rm ren}$ is to enjoy the symmetries of the Schwarzschild manifold then it must have the structure

$$\langle T_{ab}\rangle_{\rm ren}=A(r)g_{ab}+C(r)\Omega_{ab},$$

$$\langle T_{jk}\rangle_{\rm ren}=B(r)g_{jk}$$

with

$$\Omega_{ab}=\frac{32M^3}{r}e^{-r/2M}\begin{bmatrix}\cosh 2\kappa t & -\sinh 2\kappa t\\ -\sinh 2\kappa t & \cosh 2\kappa t\end{bmatrix}$$

in (u,v) coordinates and where $C(2M)$ must vanish in order that $\langle T_{ab}\rangle_{\rm ren}$ be well defined on the bifurcation two-sphere. Our calculation has yielded the values of $A(2M)$ and $B(2M)$. The term involving $C(r)$ gives a contribution to the expectation value of the stress tensor for points on the horizon away from the bifurcation two-sphere since for these points $|t|\to\infty$. We find, in fact,

$$C(r)\Omega_{ab}\to 8M^3e^{-2}C'(2M)V^2\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

for points on the future horizon and

$$C(r)\Omega_{ab}\to 8M^3e^{-2}C'(2M)U^2\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

for points on the past horizon. Unfortunately, $C'(2M)$ remains undetermined by our analysis.

V. ON THE RESPONSE OF AN UNRUH BOX

The response of an idealized "particle detector" has been considered by Unruh and has been further examined by DeWitt.[15] A black box is endowed with internal degrees of freedom which enable it to be in states corresponding to varying degrees of excitation. A concrete example of this sort of device is the one originally proposed by Unruh which consists of a nonrelativistic Schrödinger particle that is confined inside the box. If the box is at rest in a static gravitational field the regime inside will be static and the internal degrees of freedom will be able to assume various energy eigenstates.

TABLE III. Values of the β_l coefficients.

l	β_l
0	0.119377314
1	0.002779253
2	0.000144501
3	0.000018365

The internal degrees of freedom of the box are now weakly coupled to the fluctuations of the scalar field via an interaction described by adding a term

$$\mathcal{L}_{\rm int}(x)=m(x)\phi(x)$$

to the Lagrangian of the scalar field. $m(x)$ may be thought of as some sort of monopole charge which for Unruh's original box might be related to the wave function $\Psi(x)$ of the Schrödinger particle by

$$m(x)=\lambda\Psi^*(x)\hat{O}\Psi(x)$$

with $\hat{O}$ some operator and λ a small coupling constant.

The box is assumed small in comparison with the length scale associated with variations in the spacetime geometry so that $m(x)$ may be written in terms of the box's path

$$x(\tau)=(t'+\tau, r', \theta', \phi')$$

as

$$m(x)=\mu(\tau)\delta(r-r')\delta(\Omega,\Omega').$$

A short calculation analogous to that of DeWitt then reveals that the rate, as measured at the box, at which the box makes transitions from a state Ψ_0 to a state Ψ_ω which differ in energy by an amount,

$$\left(1-\frac{2M}{r'}\right)^{-1/2}\omega$$

as measured at the box, is

$$\mathcal{R}(\omega|r')=|\langle\Psi_\omega|\mu(0)|\Psi_0\rangle|^2\left(1-\frac{2M}{r'}\right)^{-1/2}\times\int_{-\infty}^{\infty}d\tau\, e^{-i\omega\tau}\langle\phi(\tau)\phi(0)\rangle, \quad (54)$$

in which $\phi(\tau)$ has been used to denote $\phi(x)$ evaluated at $x(\tau)$. An important observation to be made from (54) is that $\mathcal{R}(\omega|r)$ is essentially determined by

$$\Pi(\omega|r)=\int_{-\infty}^{\infty}d\tau\, e^{-i\omega\tau}\langle\phi(\tau)\phi(0)\rangle\,,$$

the Fourier transform of the autocorrelation function of the field.

The calculation of $\Pi(\omega|r)$ in the various regimes of interest is a straightforward matter since no renormalization is involved. The results are summarized in Table IV.

As $r\to\infty$ we see that the asymptotic forms of $\Pi(\omega|r)$ are in accord with the response expected from a "particle" detector given the interpretation that we have placed on the three vacuum states. Note, however, that the same cannot be said as $r\to 2M$. We find, for example, that in the Hartle-Hawking vacuum the response of an Unruh box becomes infinite as the horizon is approached while $\langle H|\phi^2|H\rangle_{\rm ren}$ and $\langle H|T_\mu{}^\nu|H\rangle_{\rm ren}$ retain an unremarkable appearance.

The origin of this apparent paradox lies in calling Unruh's box a particle detector. We have seen from (54) that the response of the box is determined not so much by "particles" as by the Fourier transform of the autocorrelation function of the field which, by the Weiner-Khinchin theorem, is the *spectrum of the fluctuations of the field*. In more graphic terms, Unruh's box is a "fluctuometer" rather than a particle detector and, therefore, contains information both about the fluctuations of the field *and* the motion of the box. Thus the readings of the box in the Hartle-Hawking vacuum as $r\to 2M$ need occasion no surprise since the acceleration to which the box must be subject in order to maintain it at constant radius tends to infinity as $r\to 2M$.

VI. THE RATE OF AREA DECREASE

We have argued that the Hawking and Unruh vacua give rise to stress tensors that are well behaved on the future horizon. It is of interest to note that this property combined with the divergence condition suffices to determine the rate of decrease of the area of the black hole.

The rate of area decrease is determined by the Newman-Penrose equation

$$\frac{d\rho}{dv}=\kappa\rho+\rho^2+\sigma\sigma^*+4\pi\langle T_{\mu\nu}\rangle_{\rm ren}\, l^\mu l^\nu\,, \qquad (55)$$

in which ρ is the convergence and σ the shear of the null congruence l^μ which generates the horizon. In (55), ρ^2 and $d\rho/dv$ are second-order quantities. If we neglect also the back reaction due to the radiation of gravitons (we shall return to this point) we may omit the $\sigma\sigma^*$ term. The lowest-order solution to (55) is then

$$\rho=-\frac{4\pi}{\kappa}\langle T_{\mu\nu}\rangle_{\rm ren} l^\mu l^\nu$$

with

$$\underline{l}=\frac{1}{2}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial r_*}\right),$$

and therefore

$$\langle T_{\mu\nu}\rangle_{\rm ren} l^\mu l^\nu=\tfrac{1}{4}\left(\langle T_{tt}\rangle_{\rm ren}+\langle T_{r_*r_*}\rangle_{\rm ren}+2\langle T_{tr_*}\rangle\right)$$
$$\to\langle T_{tr_*}\rangle \quad \text{as } r\to 2M\,.$$

The last line follows from the fact that we must have

$$\langle T_{tt}\rangle_{\rm ren}+\langle T_{r_*r_*}\rangle_{\rm ren}-2\langle T_{tr_*}\rangle=O((r-2M)^2)$$
$$\text{as } r\to 2M$$

in order for $\langle T_{\mu\nu}\rangle_{\rm ren}$ to be regular in a freely falling frame on the future horizon.

The divergence condition for the flux components implies that

$$T_{tr^*}=-\frac{L}{4\pi r^2}$$

with L the luminosity of the black hole at infinity.

TABLE IV. The asymptotic values of the function $\Pi(\omega|r)$ which determines the response of an Unruh box. θ denotes the step function.

	Boulware vacuum	Unruh vacuum	Hartle-Hawking vacuum
$r\to 2M$	$\dfrac{-\omega\,\theta(-\omega)}{2\pi(1-2M/r)}$	$\dfrac{\omega}{2\pi(1-2M/r)(e^{2\pi\omega/\kappa}-1)}$	$\dfrac{\omega}{2\pi(1-2M/r)(e^{2\pi\omega/\kappa}-1)}$
$r\to\infty$	$-\dfrac{\omega}{2\pi}\theta(-\omega)$	$\dfrac{1}{r^2}\dfrac{\Sigma(2l+1)\lvert B_l(\omega)\rvert^2}{8\pi\omega(e^{2\pi\omega/\kappa}-1)}-\dfrac{\omega}{2\pi}\theta(-\omega)$	$\dfrac{\omega}{2\pi(e^{2\pi\omega/\kappa}-1)}$

Now the rate of area decrease is given by

$$\frac{dA}{dv} = -2\int \rho\, dA \tag{56}$$

from which it follows that

$$\frac{dA}{dv} = -\frac{2AL}{M}$$

which is (of course) just the relation that was to be expected, since setting $A = 16\pi M^2$ this equation becomes

$$\frac{dM}{dv} = -L\,.$$

It is curious fact that one may deduce the value of $\langle T_{\mu\nu}\rangle_{\rm ren} l^\mu l^\nu$ on the horizon without actually doing any renormalization; simply the knowledge that the renormalized stress tensor is finite there suffices. From the way that $\sigma\sigma^*$ enters into the Newman-Penrose equation (55) it would seem that we must also have

$$\langle \sigma\sigma^*\rangle_{\rm ren} \to -\frac{1}{4M^2} L_{\rm gravitons}$$

as $r \to 2M$.

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APPENDIX A: ON THE ASYMPTOTIC EVALUATION OF MODE SUMS

We review here a number of techniques which prove useful in the evaluation of the results quoted in Sec. II.

Consider, for example, the quantities

$$\sum_{l=0}^{\infty} (2l+1)|\overleftarrow{R}_l(\omega|r)|^2 \tag{A1}$$

and

$$\sum_{l=0}^{\infty} (2l+1)|\overrightarrow{R}_l(\omega|r)|^2\,. \tag{A2}$$

The asymptotic forms of (A1) as $r \to 2M$ and of (A2) as $r \to \infty$ may be obtained directly by substitution of the asymptotic forms that define the radial function.

The asymptotic form of (A1) as $r \to \infty$ may be found by use of the WKB approximation. In particular, we may consider the WKB approximation to the point-separated Boulware Green's function. As $r \to \infty$ for fixed $t > 0$ we have

$$G_B(t,r,\theta,\phi;0,r,\theta,\phi) = i\int_0^\infty \frac{d\omega}{16\pi^2\omega} e^{-i\omega t} \sum_{l=0}^{\infty} (2l+1)[|\overrightarrow{R}_l(\omega|r)|^2 + |\overleftarrow{R}_l(\omega|r)|^2] \underset{r\to\infty}{\sim} \frac{-i}{4\pi^2 t^2}$$

and hence

$$\sum_{l=0}^{\infty} (2l+1)|\overleftarrow{R}_l(\omega|r)|^2 \underset{r\to\infty}{\sim} 4\omega^2\,.$$

The remaining case is the asymptotic form of (A2) as $r \to 2M$ which requires a certain amount of care. A possible line of attack is to implement the WKB approximation; however, this suffers from the disadvantage that in its naive form it is not uniformly valid with respect to l. This feature can be remedied[8] but the whole procedure becomes then somewhat ungainly, especially for fields of nonzero spin. Moreover, when dealing with fields of spin greater than zero it is all too easy to discard important terms and lose the low-frequency behavior of angular sums near the horizon.

Since no great complication is introduced thereby we shall treat the problem in rather greater generality than is in fact required and seek asymptotic solutions to the radial Teukolsky equation for the Kerr metric. The scalar radial equation for the Schwarzschild metric results from this by setting both the spin parameter s and the Kerr parameter a equal to zero. In standard notation[16] the Teukolsky equation assumes the form

$$\Delta\frac{d^2R}{dr^2} + 2(s+1)(r-M)\frac{dR}{dr} + \left\{\frac{K[K-2is(r-M)]}{\Delta} + 4i\omega s r - \lambda\right\}R = 0 \tag{A3}$$

with

$$K = (r^2+a^2)\omega - am\,.$$

If we seek solutions to (A3) as $r \to r_+$ then by setting

$$q = \frac{2}{r_+ - r_-}[\omega(r_+^2 + a^2) - am] = \frac{1}{\kappa}(\omega - m\Omega)$$

we may write

$$(r-r_+)\frac{d^2R}{dr^2}+(s+1)\frac{dR}{dr}+\left[\frac{q(q-2is)}{4(r-r_+)}-\frac{l^2}{(r_+-r_-)}\right]R=0\,. \quad \text{(A4)}$$

In the above we have replaced λ by its asymptotic form for large l since it is only when l is large that it may not be neglected in comparison with

$$\left(\frac{r-r_+}{r_+-r_-}\right)^{-1}.$$

Let us define a new radial variable by

$$\xi^2=\frac{r-r_+}{r_+-r_-}$$

in terms of which (A4) may be rewritten as

$$\frac{d^2R}{d\xi^2}+\frac{(2s+1)}{\xi}\frac{dR}{d\xi}+\left[\frac{q(q-2is)}{\xi^2}-4l^2\right]R=0\,. \quad \text{(A5)}$$

(A5) admits solutions which may be expressed in terms of modified Bessel functions as

$$\xi^{-s}K_{s+iq}(2l\xi)\,,\quad \xi^{-s}I_{-s-iq}(2l\xi)\,. \quad \text{(A6)}$$

The virtue of these solutions being that *they are uniformly valid with respect to l*.

We specialize now to the case $s=0$ and the outgoing function $\vec{R}_l(\omega|r)$ for the Schwarzschild metric and write

$$\vec{R}_l \underset{r\to 2M}{\sim} a_l K_{iq}(2l\xi)+b_l I_{-iq}(2l\xi)\,. \quad \text{(A7)}$$

We observe that as $l\to\infty$ for fixed ξ $\vec{R}_l(\omega|r)\to 0$ since r lies then in the region for which the effective potential[8] for the radial function is large. We may deduce from this that b_l is an exponentially small function of l for large l. The second term in (A7) will therefore make a contribution to the sum (A2) which remains bounded as $\xi\to 0$ and which may be neglected in comparison with that of the first term which will be of order ξ^{-2}. The coefficient a_l may be determined by taking the asymptotic limit $\xi\to 0$ for fixed l in (A7) and comparing the result with the WKB form

$$\vec{R}_l(\omega|r)\sim\frac{1}{r}e^{i\omega r*}+\vec{A}_l(\omega)\frac{e^{-i\omega r*}}{r}\,.$$

Thus we find

$$a_l=\frac{e^{iq/2}l^{-iq}}{M\Gamma(-iq)}$$

and hence to leading order

$$\sum_{l=0}^{\infty}(2l+1)|\vec{R}_l(\omega|r)|^2\sim\frac{2}{M^2\Gamma(iq)\Gamma(-iq)}\int_0^\infty dl\,l\,K_{iq}{}^2(2l\xi)=\frac{4\omega^2}{1-2M/r}\,.$$

The contribution of a mode u to the (conformal) stress tensor is

$$T_{\mu\nu}[u,u^*]=\tfrac{2}{3}u_{;\mu}u^*{}_{;\nu}-\tfrac{1}{3}u_{;\mu\nu}u^*-\tfrac{1}{6}g_{\mu\nu}u_{;\alpha}u_;{}^{*\alpha}\,.$$

Therefore, choosing modes u appropriate to the Boulware vacuum, we may write[7]

$$\langle B|T_{\mu\nu}|B\rangle=\sum_{lm}\int_0^\infty d\omega\{T_{\mu\nu}[\vec{u},\vec{u}^*]+T_{\mu\nu}[\overleftarrow{u},\overleftarrow{u}^*]\}\,,$$

$$\langle U|T_{\mu\nu}|U\rangle=\sum_{lm}\int_0^\infty d\omega\{\coth(\pi\omega/\kappa)T_{\mu\nu}[\vec{u},\vec{u}^*]+T_{\mu\nu}[\overleftarrow{u},\overleftarrow{u}^*]\}\,, \quad \text{(A8)}$$

$$\langle H|T_{\mu\nu}|H\rangle=\sum_{lm}\int_0^\infty d\omega\coth\frac{\pi\omega}{\kappa}\{T_{\mu\nu}[\vec{u},\vec{u}^*]+T_{\mu\nu}[\overleftarrow{u},\overleftarrow{u}^*]\}\,.$$

It is a straightforward matter to evaluate the mode sums involved in differences of stress tensors by employing the methods that we have discussed. As an example consider the limit as $r\to\infty$ of

$$\langle H|T_{\mu\nu}|H\rangle-\langle B|T_{\mu\nu}|B\rangle=2\sum_{lm}\int_0^\infty\frac{d\omega}{e^{2\pi\omega/\kappa}-1}\{T_{\mu\nu}[\vec{u},\vec{u}^*]+T_{\mu\nu}[\overleftarrow{u},\overleftarrow{u}^*]\}\,. \quad \text{(A9)}$$

Now the WKB approximation to the point-separated Boulware stress tensor in which the two points are taken to be (t,r,θ,ϕ) and $(t+\tau,r,\theta,\phi)$ is known,[6] in the limit $r\to\infty$ for fixed τ, to be

$$\frac{1}{2\pi^2\tau^2}\begin{pmatrix}-3&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

However, this quantity is also given by inserting a factor $e^{-i\omega\tau}$ into (A8b), which implies

$$\sum_{lm}\{T_{\mu\nu}[\vec{u},\vec{u}^*]+T_{\mu\nu}[\overleftarrow{u},\overleftarrow{u}^*]\}=\frac{\omega^3}{12\pi^2}\begin{pmatrix}-3&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

which establishes (19).

APPENDIX B: THE EVALUATION OF $\langle T_{a'}{}^{a'}\rangle_{\text{subtract}}$

We record here the values of the individual finite direction-dependent terms in (38) since these terms will be required to repeat the calculation of $\langle T_{\mu'\nu'}\rangle_{\text{ren}}$ for higher-spin fields. We shall number the terms in $\langle T_{a'}{}^{a'}\rangle_{\text{finite subtract}}$ from 1 to 11, and by "term" we shall understand a product of Riemann tensors and separation vectors with numerical coefficient unity. Thus, for example, term 7 is

$$R_{\rho'\beta'\tau'\gamma'}R^{\tau'}{}_{\alpha'}{}^{\rho'}{}_{a'}\frac{\sigma^{a'}\sigma^{\alpha'}\sigma^{\beta'}\sigma^{\gamma'}}{(\sigma_{\rho'}\sigma^{\rho'})^2},$$

which appears in the expression for $-8\pi^2\langle T_{a'}{}^{a'}\rangle_{\text{finite subtract}}$ with coefficient $-\frac{4}{45}$. With this understanding we record the values of the various terms in Table V.

The evaluation of the entries in Table V is straightforward given Eq. (39) with the possible exception of term 9 which involves the second covariant derivative of the Riemann tensor. In order to evaluate this term we set

$$g_{\mu\nu}=p_{\mu\nu}+q_{\mu\nu}$$

with

$$p_{\mu\nu}=\begin{bmatrix} g_{ab} & 0\\ 0 & 0\end{bmatrix}\quad\text{and}\quad g_{\mu\nu}=\begin{bmatrix} 0 & 0\\ 0 & g_{jk}\end{bmatrix}$$

and first establish as a lemma that

$$p_{\mu\nu;a}=q_{\mu\nu;a}=0 \tag{B1}$$

and

$$p_{\mu\nu;ab}=q_{\mu\nu;ab}=0\,. \tag{B2}$$

To establish the lemma we note the following identities:

$$q_{\mu\nu;a}=r_{,a}\frac{\partial}{\partial r}q_{\mu\nu}-\Gamma_{\mu a}{}^{i}q_{i\nu}-\Gamma_{\nu a}{}^{i}q_{\mu i}\,,$$

$$\frac{\partial}{\partial r}q_{\mu\nu}=\frac{2}{r}q_{\mu\nu}\,,$$

$$\Gamma_{\mu a}{}^{i}=q^{i}{}_{\mu}\frac{r_{,a}}{r}\,,$$

the last of these is true in Schwarzschild coordinates and follows from (32). Thus

$$q_{\mu\nu;a}=\frac{2r_{,a}}{r}q_{\mu\nu}-\frac{r_{,a}}{r}(q^{i}{}_{\mu}q_{i\nu}+q^{i}{}_{\nu}q_{i\mu})=0\,.$$

Since, trivially, $p_{\mu\nu;\alpha}=-q_{\mu\nu;\alpha}$, we have established (B1).

Now

$$\begin{aligned}p_{\mu\nu;ab}&=(p_{\mu\nu;a})_{,b}-\Gamma_{\mu b}{}^{\kappa}p_{\kappa\nu;a}\\&\quad-\Gamma_{\nu b}{}^{\kappa}p_{\mu\kappa;a}-\Gamma_{ab}{}^{\kappa}p_{\mu\nu;\kappa}\\&=-\Gamma_{ab}{}^{\kappa}p_{\mu\nu;\kappa}\\&=0\end{aligned}$$

since by (32), $\Gamma_{ab}{}^{k}=0$, and this establishes (B2).

We are interested in the quantity

$$\begin{aligned}p^{\mu\nu}R_{\mu\alpha\nu\beta;\gamma\delta}\sigma^{\alpha}\sigma^{\beta}\sigma^{\gamma}\sigma^{\delta}&=\sigma^{\alpha}\sigma^{\beta}\sigma^{\gamma}\sigma^{\delta}[(p^{\mu\nu}R_{\mu\alpha\nu\beta})_{;\gamma\delta}-R_{\mu\alpha\nu\beta}p^{\mu\nu}{}_{;\gamma\delta}\\&\quad-R_{\mu\alpha\nu\beta;\gamma}p^{\mu\nu}{}_{;\delta}-R_{\mu\alpha\nu\beta;\delta}p^{\mu\nu}{}_{;\gamma}]\,.\end{aligned}$$

Now

$$p^{\mu\nu}{}_{;\gamma\delta}\sigma^{\gamma}\sigma^{\delta}=0\text{ and }p^{\mu\nu}{}_{;\gamma}\sigma^{\gamma}=0$$

in virtue of the lemma. We also have

$$p^{\mu\nu}R_{\mu\alpha\nu\beta}=\frac{2M}{r^3}(p_{\alpha\beta}-q_{\alpha\beta})$$

and the required result follows from an elementary calculation.

APPENDIX C: THE NUMERICAL EVALUATION OF THE β_l COEFFICIENTS

It follows from (14) and (15) that the Wronskian relation between the radial functions p_l^1 and q_l^1 takes the form

$$p_l^1(\eta)\frac{d}{d\eta}q_l^1(\eta)-q_l^1(\eta)\frac{d}{d\eta}p_l^1(\eta)=-\frac{2}{\eta^2-1}\,.$$

If we divide this equation by $(p_l^1)^2$, integrate and recall the boundary condition that $q_l^1(\eta)\to 0$ as $\eta\to\infty$, we obtain an integral representation of q_l^1 in terms of p_l^1

$$q_l^1(\eta)=2p_l^1(\eta)\int_\eta^\infty\frac{d\xi}{(\xi^2-1)[p_l^1(\xi)]^2}\,. \tag{C1}$$

Now, by obtaining a series solution to (14), it is easy to show that as $\xi\to 1$

$$\frac{1}{[p_l^1(\xi)]^2}=\frac{1}{(\xi-1)}-\frac{l(l+1)}{2}+O(\xi-1)\,. \tag{C2}$$

We now rewrite (C1) in the form

TABLE V. The values of the individual terms needed for the computation of $-8\pi^2\langle T_{a'}{}^{a'}\rangle_{\text{finite subtract}}$.

Term number	1	2	3	4	5	6	7	8	9	10	11
Coefficient	$\frac{1}{360}$	$\frac{1}{45}$	$\frac{1}{45}$	$\frac{1}{45}$	$-\frac{1}{45}$	$-\frac{1}{180}$	$-\frac{4}{45}$	$-\frac{1}{90}$	$\frac{1}{90}$	$-\frac{1}{45}$	$\frac{1}{36}$
$M^4\times$ (term)	$\frac{3}{4}$	$\frac{1}{8}$	$\frac{3}{32}$	$\frac{5}{32}$	$\frac{3}{16}$	0	$\frac{3}{32}$	$-\frac{3}{16}$	$-\frac{3}{32}$	$\frac{1}{16}$	$\frac{3}{16}$

$$q_l^1(\eta)=2p_l^1(\eta)\times\int_\eta^\infty \frac{d\xi}{(\xi^2-1)}\left(\frac{1}{[p_l^1(\xi)]^2}-\frac{1}{(\xi-1)}+\frac{l(l+1)}{2}\right)+2p_l^1(\eta)\int_\eta^\infty \frac{d\xi}{(\xi^2-1)}\left(\frac{1}{\xi-1}-\frac{l(l+1)}{2}\right) \quad \text{(C3)}$$

and observe that the second integral can be evaluated exactly and that in virtue of (C2) the first integral converges as $\eta\to 1$.

From (C3) we obtain an asymptotic expansion of $q_l^1(\eta)$ as $\eta\to 1$,

$$q_l^1(\eta)=(\eta-1)^{-1/2}+\tfrac{1}{2}[l(l+1)+1](\eta-1)^{1/2}\ln(\eta-1)+\{2I_l+\tfrac{1}{4}l(l+1)-\tfrac{1}{2}[l(l+1)+1]\ln 2\}(\eta-1)^{1/2}+O((\eta-1)^{3/2}\ln(\eta-1)), \quad \text{(C4)}$$

where

$$I_l=\int_1^\infty \frac{d\xi}{(\xi^2-1)}\left\{\frac{1}{[p_l^1(\xi)]^2}-\frac{1}{(\xi-1)}+\frac{l(l+1)}{2}\right\}. \quad \text{(C5)}$$

Alternatively, we may expand the relation (45) near $\eta=1$ to obtain

$$q_l^1(\eta)=(\eta-1)^{-1/2}+\tfrac{1}{2}[l(l+1)+1](\eta-1)^{1/2}\ln(\eta-1)+\{[l(l+1)+1][\psi(l+1)+\gamma-\tfrac{1}{2}\ln 2-\tfrac{1}{2}]+\tfrac{1}{4}+\beta_l\}(\eta-1)^{1/2}+O((\eta-1)^{3/2}\ln(\eta-1)), \quad \text{(C6)}$$

where

$$\psi(z)=\frac{d}{dz}\ln\Gamma(z)$$

and γ is Euler's constant.

Comparison of (C4) and (C6) provides us with an expression for the β_l that is amenable to numerical computation:

$$\beta_l=2I_l-[l(l+1)+1][\psi(l+1)+\gamma-\tfrac{3}{4}]-\tfrac{1}{2}.$$

The evaluation of (C5) is conveniently performed by dividing the range of integration into the two ranges $1\le\xi\le 2$ and $2\le\xi<\infty$. In the former range the function p_l^1 is rapidly calculated by summing a series solution to (14). The p_l^1 was evaluated in the latter range by a fourth-order Runge-Kutta scheme to integrate (14) using as starting values the $p_l^1(2)$ and $p_l^{1\prime}(2)$ calculated from the series solution. The advantage of basing the calculation on the numerical integration of (14) for p_l^1 rather than attempting a direct numerical integration of the q_l^1 is that it is difficult to perform the latter in a stable manner. It was found that a programmable calculator sufficed to determine the β_l to nine decimal places.

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