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Quantum fields in curved space–times

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The problem of obtaining a quantum description of the (real) Klein–Gordon system in a given curved space–time is discussed. An algebraic approach is used. The $*$-algebra of quantum operators is constructed explicitly and the problem of finding its $*$-representation is reduced to that of selecting a suitable complex structure on the real vector space of the solutions of the (classical) Klein–Gordon equation. Since, in a static space-time, there already exists, a satisfactory quantum field theory, in this case one already knows what the 'correct' complex structure is. A physical characterization of this 'correct' complex structure is obtained. This characterization is used to extend quantum field theory to non-static space-times. Stationary space–times are considered first. In this case, the issue of extension is completely straightforward and the resulting theory is the natural generalization of the one in static space–times. General, non-stationary space-times are then considered. In this case the issue of extension is quite complicated and we only present a plausible extension. Although the resulting framework is well-defined mathematically, the physical interpretation associated with it is rather unconventional. Merits and weaknesses of this framework are discussed.

1. Introduction

The purpose of this paper is to suggest a mathematical and conceptual framework for describing quantum fields in curved space–times.

Fix a space–time (M, g_{ab}).[1]† For simplicity, we shall restrict ourselves to (the minimally coupled) Klein–Gordon fields on this space–time, i.e. to scalar fields Φ on M which satisfy $\nabla^a \nabla_a \Phi - \mu^2 \Phi = 0$, where ∇_a is the derivative operator on (M, g_{ab}) and μ is a fixed real number. To obtain a quantum description of these fields, we must construct a Hilbert space of states, a $*$-algebra of operators, and introduce a rule for dynamics. As is now well known, the construction of the $*$-algebra (of field operators) itself is quite straightforward. Hence, it is convenient to use an algebraic approach; i.e. to first construct, abstractly, the $*$-algebra, and then try to obtain the Hilbert space by choosing an appropriate representation of this $*$-algebra. The essential difficulty arises, of course, due to the lack of a suitable prescription for

† For notes 1–30, see pp. 392–4.

making this choice in a general curved space–time. It is only in the case when the underlying space–time is static that such a prescription has been available: in this case, one can define creation and annihilation operators by decomposing field operators into positive and negative frequency parts, define the vacuum state using the annihilation operators, and construct the required $*$-representation using this vacuum state. Here, the essential simplification arises because (using the presence of the Killing vector) one can decompose fields into positive and negative frequency parts in a natural way. Hence, to extend this quantum description to non-static contexts one often *begins* by asking the following question: how would one decompose fields into their positive and negative frequency parts on such space-times?

Perhaps the most important step in the present paper is that we begin by asking a different – and, at first sight, what appears to be just the opposite – question: Why does this procedure involving decomposition of fields 'work' in the first place? From a physical view-point, there appears to be no *a priori* reason as to why one should have to use a 'global' operation such as this decomposition to obtain a mathematical description of such (intuitively) 'local' entity as a particle. Hence, we begin by re-examining the static case itself. We find that there exist certain physically motivated requirements on the required $*$-representation which do in fact single out, among all possible $*$-representations, the one obtained by using the positive and negative frequency decomposition. One might therefore say that these restrictions constitute the 'physics' behind the use of this decomposition. We adopt the view-point that *it is these restrictions which are 'fundamental' and that the positive and negative frequency decomposition, available in the static case, is only a mathematically convenient way of incorporating them.* The idea then is to use the appropriately modified versions of these restrictions to select the required $*$-representations in non-static space-times.

In § 2 we begin with an explicit construction of the $*$-algebra $\mathscr{A}$ of field-operators. Then, we investigate the possible $*$-representations of this $\mathscr{A}$. By imposing three rather general and physically motivated requirements, we reduce the freedom available in selecting the appropriate representation to that of selecting a suitable complex structure[2] J on the vector space of (real) solutions of the Klein–Gordon equation. Thus, in any given curved space-time, the problem of obtaining the required $*$-representation is reduced to that of choosing an appropriate J.

In § 3 we consider static space–times.[3] In this case, the presence of a natural decomposition of fields into positive and negative frequency parts provides us with a natural complex structure J. Furthermore, this decomposition plays no other role in the resulting quantum description: it is required *only* in order to obtain this J. Hence, we focus our attention on this J and try to understand, from a physical view-point, why this J is the 'correct' one. That is to say, we try to obtain a physical characterization of this J. We find that there is in fact a physical requirement which singles out this J among all possible complex structures. This requirement – henceforth called the energy requirement – essentially says that the energy of each one particle state (in the resulting $*$-representation) be equal to that of the corresponding classical field.

We then adopt the attitude that this 'energy requirement' is somehow fundamental to quantum field theory and try to extend it to non-static space–times. In §4, we consider stationary[4] space–times. In a general stationary space–time, although one can introduce the notion of positive and negative frequency fields using the Killing vector, there is apparently no natural prescription for *decomposing* fields into their positive and negative frequency parts. It is this absence of prescription that has prevented a satisfactory extension of quantum field theory to stationary space–times (see, for example, Fulling 1972). It turns out, however, that the 'energy requirement' does admit a straightforward extension to the stationary case. Furthermore, it again selects for us, as in the static case, a unique complex structure. Thus, we do in fact obtain the required extension of quantum field theory without having to first decompose fields into their positive and negative frequency parts.

In §5, we consider space–times which are not necessarily stationary. As one might expect, the issue of the extension of the 'energy requirement' is quite complicated in this case. We just select a plausible extension and proceed, as before, to construct the required $*$-representation. The final result is one possible mathematical and conceptual framework for describing quantum fields. Although the mathematical aspects of this framework are all well defined, the physical interpretation which emerges out of it is quite unconventional in certain respects. Hence, the discussion of this section is to be regarded more as a preliminary exploration for obtaining a quantum theory of fields in general curved space–times, than as a completed theory in itself.

2. General framework

The discussion of this section[5] is divided into two parts: we first construct the $*$-algebra of field operators and then discuss the issue of obtaining the appropriate Hilbert space of states.

The $*$-algebra of quantum operators

There are several ways in which one can construct this $*$-algebra. Here we summarize a procedure which is a slightly modified version of the one discussed by Segal (1967) in the context of Minkowski space.

Let V denote the vector-space of all well-behaved[6] real-valued solutions of the Klein–Gordon equation on the given space–time. With each element Φ of V, associate an abstract operator $F(\Phi)$. We shall call this operator a *field-operator*. Consider the free $*$-algebra generated by these field operators. On this $*$-algebra, impose[7] the following conditions:

self-adjointness: $$F(\Phi) = F^*(\Phi), \quad (1)$$

linearity: $$F(\Phi_1 + r\Phi_2) = F(\Phi_1) + rF(\Phi_2), \quad (2)$$

commutation relations: $$[F(\Phi_1), F(\Phi_2)] = \left\{ \mathrm{i} \int_{t_0} (\Phi_2 \nabla_a \Phi_1 - \Phi_1 \nabla_a \Phi_2)\, \mathrm{d}S^a \right\} I. \quad (3)$$

Here, Φ, Φ_1 and Φ_2 are arbitrary elements of V; r is an arbitrary real number; and t_0 a space-like Cauchy surface in (M, g_{ab}).[8] Denote the resulting $*$-algebra by $\mathscr{A}$. This $\mathscr{A}$ is the required $*$-algebra of quantum operators.

Note that the integral on the right side of equation (3) is linear and anti-symmetric in Φ_1 and Φ_2. Hence, it defines a second rank, skew tensor, say Ω, on the vector-space V: $\Omega(\Phi_1, \Phi_2) = \int_{t_0} (\Phi_2 \nabla_a \Phi_1 - \Phi_1 \nabla_a \Phi_2)\, \mathrm{d}S^a$. This Ω is in fact the tensor which is induced on V by the natural symplectic structure on the classical phase-space of the Klein–Gordon system. Thus, by imposing condition (3) above, we have only required that 'the commutator between any two field operators be i times the Poisson bracket between their classical analogues.'

Finally, we show that the $*$-algebra $\mathscr{A}$, constructed above, is in fact naturally isomorphic to the more familiar $*$-algebra of the canonically conjugate operators. Every solution Φ of the classical Klein–Gordon equation can be characterized by a pair, (f, g) of functions on an arbitrarily chosen (but fixed) space-like Cauchy surface t_0: $f = \Phi|_{t_0}$, and $g = n^a \nabla_a \Phi|_{t_0}$, where n^a is the unit future-directed normal to t_0. Denote $F(0, g)$ by $\Phi(g)$ and $F(f, 0)$ by $\Pi(f)$. Then, it follows from equations (1)–(3) that (i) these Φ's and Π's are self-adjoint; (ii) they are (real) linear in their arguments; and (iii) they satisfy the canonical commutation relations

$$[\Phi(g), \Phi(g')] = 0, \quad [\Pi(f), \Pi(f')] = 0 \quad \text{and} \quad [\Phi(g), \Pi(f)] = \mathrm{i} \int_{t_0} fg \, \mathrm{d}V,$$

where $\mathrm{d}V$ is the natural volume element, induced by the space–time metric g_{ab}, on t_0. Thus, $\mathscr{A}$ is indeed a copy of the more familiar $*$-algebra generated by the (smeared out) field-configuration and field-momentum operators.

The Hilbert space of quantum states

Having constructed the $*$-algebra $\mathscr{A}$, we can now face the more difficult issue of obtaining the appropriate Hilbert space of quantum states.

To obtain this Hilbert space, we must construct a $*$-representation[9] of $\mathscr{A}$. As is well known (see, for example, Wightman 1967), the existence of the infinitely many degrees of freedom in the Klein–Gordon system leads to the existence of several inequivalent $*$-representations of this $\mathscr{A}$. Hence, we must impose some physical requirements to single out the appropriate one. First, we shall demand that the space of all quantum states (i.e. the Hilbert space underlying our $*$-representation) be a symmetric Fock-space $\mathscr{F}$, based on some Hilbert space $\mathscr{H}$. As a consequence of this requirement, a 'particle interpretation' would be naturally built into the formalism: an element of $\mathscr{H}$ could be thought of as representing a quantum state of a single particle and that of $\mathscr{F}$, as representing some superposition of n-particle states for various values of n. Secondly, we need a requirement about 'the size' of the one-particle Hilbert space $\mathscr{H}$. An examination of the standard quantum field theory in Minkowski space suggests the appropriate condition;[10] we shall demand that, *as a real vector-space*, $\mathscr{H}$ be a copy of the vector-space V of all well-behaved, real-valued solutions of the classical Klein–Gordon equation. Thirdly, we shall require that the $*$-representation be such that each (abstractly defined) field operator is mapped to the usual sum of the (concretely defined) creation and annihilation operators on the Fock-space $\mathscr{F}$. This requirement reflects the intuitive idea that a quantum field is to be thought of as 'an assembly of (quantized) simple harmonic oscillators.' (All these requirements are more or less 'universal'; they are invariably imposed, although often implicitly in most discussions of quantum fields in curved space–times.) We shall now show that these three requirements restrict the admissible representations to a rather small class. Furthermore, we shall be able to so arrange our formalism that the resulting restricted freedom is not only explicitly exhibited but its mathematical investigation is also made easy.

Consider the one-particle Hilbert space $\mathscr{H}$. As a real vector space, this $\mathscr{H}$ is to be a copy of V. However, since the elements of $\mathscr{H}$ are to represent quantum states of a particle, $\mathscr{H}$ must have, in addition, the structure of a complex Hilbert space. Hence, we must introduce on V, a complex structure J, and define, on the resulting complex vector space (V, J), a hermitian inner product $\langle , \rangle$. The one-particle Hilbert space $\mathscr{H}$ would then be the Cauchy completion of the complex inner-product space $(V, J, \langle , \rangle)$. Choose a pair $(J, \langle , \rangle)$. We shall now show that the three requirements, made above, not only determine a unique representation of $\mathscr{A}$ based on this pair, but also provides a restrictive compatibility requirement on the pair itself.

The space of all quantum states is, by the first requirement, the symmetric Fock–space $\mathscr{F}$ based on the Hilbert space $\mathscr{H}$: $\mathscr{F} = \bigoplus_{k=0}^{\infty} S\mathscr{H}^{\otimes k}$, where $S\mathscr{H}^{\otimes k}$ is the Hilbert space of all kth rank symmetric tensors over $\mathscr{H}$, and where $\oplus$ denotes the operation of taking direct sums (of Hilbert spaces). Thus, a typical element Ψ of $\mathscr{F}$ is a string with a finite number of non-zero entries $\Psi = (\xi^0, \xi^1, \xi^2, \ldots, \xi^k, \ldots)$, where ξ^k is in $S\mathscr{H}^{\otimes k}$. On this $\mathscr{F}$ there exist naturally defined creation and annihilation operators. Associated with each σ^1 in $\mathscr{H}$ we have

$$C(\sigma^1)\Psi = (0, \xi\,\sigma^1, \sqrt{(2)}\xi^1 \cap \sigma^1, \sqrt{(3)}\xi^2 \cap \sigma^1, \ldots), \tag{4}$$

$$A(\sigma^1)\Psi = (\overline{\sigma^1} \ulcorner \xi^1, \sqrt{(2)}\,\overline{\sigma^1} \ulcorner \xi^2, \sqrt{(3)}\,\overline{\sigma^1} \ulcorner \xi^3, \ldots), \tag{5}$$

where $\cap$ denotes the operation of taking symmetrized outer products; $\ulcorner$ the usual contraction defined by the hermitian metric on $\mathscr{H}$; and where $\overline{\sigma^1}$ is the complex-conjugate of σ^1 in $\mathscr{H}$. Thus, one may regard $C(\sigma^1)$ as 'creating a particle in the state σ^1' and $A(\sigma^1)$ as 'annihilating a particle in the state σ^1'. Following properties of these operators result directly from the above definitions: (1) $A(\sigma^1)$ is the adjoint of $C(\sigma^1)$; (ii) each creation (annihilation) operator is complex linear (anti-linear) in its argument; and (iii)

$$[C(\sigma^1)\,C(\tau^1)] = 0, \quad [A(\sigma^1), A(\tau^1)] = 0 \quad \text{and} \quad [A(\sigma^1), C(\tau^1)] = \langle \sigma^1, \tau^1 \rangle I,$$

where $\langle \sigma^1, \tau^1 \rangle$ is the inner-product between the one particle states σ^1 and τ^1, and where I is the identity operator on $\mathscr{F}$ (for details, see Geroch 1971). We now use our third requirement: we demand that every (abstract) field-operator $F(\Phi)$ be represented by the concrete operator $C(\Phi^1)+A(\Phi^1)$ on $\mathscr{F}$.[11] It is this condition which gives rise to the compatibility requirement on the pair $(J, \langle,\rangle)$ with which we began. It follows from the properties (i) and (ii) (listed above) of creation and annihilation operators that the operator $C(\Phi^1)+A(\Phi^1)$ is self-adjoint and real-linear in its argument Φ^1. Thus, the identification of $C(\Phi^1)+A(\Phi^1)$ with (the representative on $\mathscr{F}$ of the abstract operator) $F(\Phi)$ is already consistent with equations (1) and (2) satisfied by the field-operators. We now consider equation (3); i.e. the commutation relation between field operators. This equation, together with the identification of $F(\Phi)$ with $C(\Phi^1)+A(\Phi^1)$, implies that we must have

$$[A(\Phi^1)+C(\Phi^1), A(\tilde{\Phi}^1)+C(\tilde{\Phi}^1)] = \mathrm{i}\Omega(\Phi,\tilde{\Phi})\, I$$

for all Φ and $\tilde{\Phi}$ in V, and hence, that $2\,\mathrm{Im}\,\langle\Phi^1,\tilde{\Phi}^1\rangle = \Omega(\Phi,\tilde{\Phi})$. This condition on the imaginary part of the inner-product $\langle,\rangle$ in turn restricts the choice of J: since $\langle,\rangle$ is a hermitian inner-product on (V, J), we must have

$$\gamma = \Omega \ulcorner J \quad \text{is a positive-definite metric on } V, \tag{6}$$

and

$$\langle\Phi^1,\tilde{\Phi}^1\rangle = \tfrac{1}{2}\gamma(\Phi,\tilde{\Phi})+\tfrac{1}{2}\mathrm{i}\Omega(\Phi,\tilde{\Phi}) \tag{7}$$

for all Φ and $\tilde{\Phi}$ in V.[12] Thus, the inner-product $\langle,\rangle$ is completely determined by the complex structure J. Furthermore, the choice of J is itself restricted: J must be compatible with the symplectic structure Ω in the sense of (6).

To summarize, in order to obtain an 'admissible' $*$-representation of $\mathscr{A}$ (i.e. a representation satisfying our three requirements), we must select, on V, a complex structure J which is compatible with the symplectic structure Ω thereon. Furthermore, the choice in this selection is all the freedom that we have: there are exactly as many 'admissible' representations as there are possible choices of J.

The rest of this paper is concerned with the issue of selecting appropriate complex structures in various contexts. To conclude the present section, we shall point out some general qualitative features of this issue. Note, first, that a quantum field theory in a given curved space–time, may be thought of as a description of the (quantum) interactions of fields and particles with the given, external gravitational field. Hence, one expects in analogy with electrodynamics, that, in a general, non-stationary space–time, there would occur processes such as spontaneous creation and annihilation of physical particles. How would such processes be described in the present framework? We shall see that these processes will make their appearance by 'forcing a time-dependence on the complex structure J'.[13] In fact, 'the rate of change of J with respect to time' will turn out to be a measure of the 'rate of particle production' by the given gravitational field. In general, it is only in the case when the underlying space-time happens to be stationary, that we shall have a canonical, time-independent complex structure, and then, there will be no particle production.

3. Static space–times

In the previous section, we reduced the problem of obtaining a quantum description of fields in any curved space–time to that of introducing a suitable complex structure J on the space of solutions of the Klein–Gordon equation on that space–time. Unfortunately, there exist several different possible choices of J, and hence one is still faced with the problem of obtaining the 'correct' one. Recall, however, that, in the case when the underlying space–time happens to be static, there exists, already, a well established and a completely satisfactory quantum description of fields (see, for example, Fulling 1972). Hence, in this case, one does know what the 'correct' complex structure is. The idea now is to use this information to obtain a prescription for selecting the required J in more general contexts. More precisely, we shall now characterize this 'correct J' in a way which can be readily generalized to non-static contexts.

Let (M, g_{ab}, t^a) be a static[3] space–time. Denote, as before, the vector space of all real, well-behaved solutions of the Klein–Gordon equation by V. We first recall the standard procedure by which one obtains the required complex structure on this V. Using the fact that t^a is a time-like hypersurface-orthogonal Killing field, one first decomposes every real solution Φ into positive and negative frequency parts $\Phi = \Phi^+ + \Phi^-$, and then sets $J\Phi = \mathrm{i}\Phi^+ + (-\mathrm{i})\,\Phi^-$. Note that, although Φ is a real solution of the Klein–Gordon equation, Φ^+ and Φ^- are themselves complex solutions (Φ^- being the complex-conjugate of Φ^+), while $J\Phi$ is again a real solution. It is straightforward to check that this J is in fact a complex structure and that it is compatible with the natural symplectic structure Ω on V. Unfortunately, however, this particular definition of J cannot be carried over to more general contexts, since it uses both the Killing property and the hypersurface orthogonality of t^a. Moreover, as was pointed out in §1, the physical motivation behind this particular construction of J is unclear already in the static case. Hence, we shall now introduce a different procedure for obtaining this same J; a procedure which, in a certain sense, brings out 'the physics' behind this choice of J. Furthermore, it will turn out that this procedure can in fact be carried over to more general contexts.

Note, first, that, if we only require that the complex structure J be compatible with the natural symplectic structure Ω, then there exist, even in the static case, several possible choices of J. Hence, to single out the one which we already know to be the correct one, we must impose some additional physical requirements. One such requirement – and, in a sense, the most natural one – is suggested by the following considerations. Recall that, every solution Φ of the classical Klein–Gordon equation plays a dual role in the present formalism: it may be regarded as a classical field on the space–time, or alternatively (for any given choice of the complex structure J), it may be regarded as an element of the one particle Hilbert space $\mathscr{H}$. This fact may be regarded as a reflexion, in the framework of quantum field theory, of the intuitive notion of the wave-particle duality. Given a solution Φ of the classical Klein–Gordon equation, one may regard it as a classical field, compute the stress-energy tensor field $T_{ab} = \nabla_a \Phi \nabla_b \Phi - \frac{1}{2} g_{ab} (\nabla^c \Phi \nabla_c \Phi + \mu^2 \Phi^2)$, and evaluate the energy associated with this Φ: $E = \int_{t_0} T_{ab} t^a \, dS^b$. (Here, t_0 is any space-like Cauchy surface in (M, g_{ab}).) On the other hand (for any given J) one may consider the one particle state Φ^1 defined by this solution, and compute from it another number, $\langle \Phi^1, H\Phi^1 \rangle$, the expectation-value of the one-particle Hamiltonian H. (H is that operator which generates translations along the integral curves of t^a. Thus,[14] $H\Phi^1 = -J(\mathscr{L}_t \Phi)^1$.) This expectation-value may be regarded as the energy associated with the one particle state Φ^1. It seems natural to require that the two energies be equal; i.e., that $\int_{t_0} T_{ab} t^a \, dS^b = \langle \Phi^1, H\Phi^1 \rangle$ for every Φ in V. Note that the left side of this equation can be computed purely classically, i.e., that it is independent of the choice of J. The right side, on the other hand, depends on this choice quite crucially. Hence, the above 'energy requirement' is a constraint on the choice of allowable J's. It turns out that this requirement in fact exhausts all the freedom that one has in this choice.

THEOREM. *There is a unique complex structure J on V such that,*

(i) *$\Omega \ulcorner J$ is a positive definite metric on V* (i.e. J is compatible with the sym plectic structure Ω); and

(ii) $\langle \Phi^1, H\Phi^1 \rangle = \int_{t_0} T_{ab} t^a \, dS^b$ *for every Φ in V* (i.e. J satisfies the energy requirement).

Furthermore, this J is precisely the one which is usually defined by $J\Phi = \mathrm{i}\Phi^+ - \mathrm{i}\Phi^-$, where Φ^+ and Φ^- are, respectively, the positive and negative frequency parts of Φ.

We now sketch the proof.[15] Using the expression for the inner-product $\langle , \rangle$ in terms of Ω and J, one obtains, from the energy requirement,

$$\Omega(\Phi, \mathscr{L}_t \Phi) = 2 \int_{t_0} T_{ab} t^a \, dS^b \tag{8}$$

and

$$\Omega(\Phi, J\mathscr{L}_t \Phi) = 0 \tag{9}$$

for every Φ in V. The first of these equations has no bearing on the choice of J. A straightforward calculation shows that this equation is always satisfied.[16] It is the second equation that restricts the possible choices of J. Starting with the most general linear operator J on V, and demanding that it be a complex structure compatible with Ω and satisfy (9), one can easily show that this J must be the standard complex structure. The proof of this assertion, although quite straightforward, is rather long and technical. It is given in the appendix in a slightly more general context.

Thus, the energy requirement provides us, in the static case, a mathematical characterization of the complex structure which we already know to be the 'correct one'. In § 4, we shall generalize this condition to the case when the underlying space-time is stationary, and in § 5, to the case when it is arbitrary.

4. Stationary space–times

We now consider the case when the underlying space–time is stationary rather than static. Thus, the Killing vector field t^a is now assumed only to be everywhere time-like and is not necessarily hypersurface orthogonal. In this case, although the notion of positive and negative frequency solutions is itself well-defined, there is no obvious natural prescription for *decomposing* a real solution into its positive and negative frequency parts. (The standard procedure used in the static case makes a crucial use of the hypersurface–orthogonality of the static Killing field, and hence, does not admit a straightforward extension to the present case.) Hence, one must select the required complex structure by some other procedure. It is here that our previous characterization of the complex structure begins to play a crucial role: we shall see that the energy-requirement (based again on the intuitive notion of the wave-particle duality) selects for us a complex structure J, and hence a Fock representation of $\mathscr{A}$, even in the stationary case.

Note first that it is meaningful to impose the energy-requirement in a general stationary space–time: $E = \int_{t_0} T_{ab}\, t^a \,\mathrm{d}S^b$ is again a conserved quantity, the energy associated with a classical field; and H defined by $H\Phi^1 = -J(\mathscr{L}_t \Phi)^1$ is again a well-defined operator on the one-particle wave-functions, the generator of translations along the integral curves of t^a. Hence, we may again demand that

$$E = \langle \Phi^1, H\Phi^1 \rangle$$

for all Φ in V. That is, to say, we may again require that our complex structure be so chosen that the energy of each one particle state equals that of the corresponding classical field. It turns out that this requirement does select for us a unique complex structure J in a general stationary space-time.

We shall begin by writing down an explicit expression for this J, especially since such an expression has not been spelled out in the context of general (i.e. non-static) stationary space-times. For this purpose it is convenient to proceed as follows. Introduce on V the following inner-product (,):

$$(\Phi, \tilde{\Phi}) := \frac{1}{2}\int_{t_0} [\nabla_a \Phi \nabla_b \tilde{\Phi} + \nabla_a \tilde{\Phi} \nabla_b \Phi - g_{ab}(\nabla^c \Phi \nabla_c \tilde{\Phi} + \mu^2 \Phi \tilde{\Phi})]\, t^a \,\mathrm{d}S^b, \qquad (10)$$

where t_0 is any space-like Cauchy surface. It is easy to verify that this (,) is well defined i.e. that the integral on the right side of equation (10) is independent of the particular choice of the surface t_0 made in its evaluation; that it is linear in each factor; that it is symmetric; and that the resulting norm is positive-definite. (In fact, (Φ, Φ) is just the energy associated with the classical field Φ.) On the resulting inner-product space $(V, (,))$, $\mathscr{L}_t$ is an anti-sy mmetric operator:

$$(\Phi, \mathscr{L}_t \tilde{\Phi}) = -(\mathscr{L}_t \Phi, \tilde{\Phi}).$$

Hence, on the Hilbert space $\bar{V}$ obtained by the Cauchy completion of $(V,(,))$, $\Theta = -\mathscr{L}_t\mathscr{L}_t$ is a positive-definite self-adjoint operator.[17] Hence, we can define on $\bar{V}$, the operator $\Theta^{-\frac{1}{2}}$ which is also positive-definite and self-adjoint. Set

$$J = \Theta^{-\frac{1}{2}}\mathscr{L}_t.^{18}.$$

It is straightforward to check that this J is a complex structure compatible with the symplectic structure Ω, and that it satisfies the energy requirements.[19].

We next show that this J is in fact the only complex structure which satisfies our requirements. Let $\hat{J}$ be another such complex structure. Set $K = -\hat{J}J$. We shall show that K must be the identity operator on $\bar{V}$, and hence $\hat{J}$ must equal J. Using the assumption that $\hat{J}$ is compatible with Ω, it is straightforward to show that K is a self-adjoint positive-definite operator on $\bar{V}$. From the assumption that $\hat{J}$ satisfies the energy requirement, it then follows that K must commute with J. Hence $K^2 = -\hat{J}JK = -\hat{J}KJ = +\hat{J}\hat{J}JJ = +I$, where I is the identity operator on $\bar{V}$. The required result then follows from the fact that K is a positive-definite self-adjoint operator on $\bar{V}$.

Thus, the energy requirement selects for us a preferred complex structure, and hence a preferred Fock representation of the $*$-algebra $\mathscr{A}$ in any given stationary space–time.

Finally, we discuss dynamics. It turns out to be convenient to work in the Heisenberg picture. The idea is to define quantum dynamics by requiring that every field-operator should mimic the time-evolution of its classical counterpart: we set $[F(\Phi)]^{\cdot} = F(\mathscr{L}_t\Phi)$. Since creation and annihilation operators can be expressed in terms of field-operators,

$$C(\Phi^1) = (1/2\mathrm{i})\,(\mathrm{i}F(\Phi)+F(J\Phi)) \quad \text{and} \quad A(\Phi^1) = (1/2\mathrm{i})\,(\mathrm{i}F(\Phi)-F(J\Phi))$$

and since J is time-independent (i.e. since $\mathscr{L}_tJ(\Phi) = J(\mathscr{L}_t\Phi)$), it follows that $[C(\Phi^1)]^{\cdot} = C((\mathscr{L}_t\Phi)^1)$ and $[A(\Phi^1)]^{\cdot} = A((\mathscr{L}_t\Phi)^1)$. Thus, under time-evolution, there is no 'mixing' between creations and annihilation operators. (Note, incidently, that if J were time-dependent, creation operators would pick up an annihilation part, and annihilation operators, a creation part, under time-evolution, i.e. there would in fact occur a 'mixing'.) Hence, it follows that the vacuum expectation value of any number-operator is zero at all times. Thus, if the underlying space–time admits a everywhere time-like Killing field, the vacuum state is indeed stable and phenomena such as the spontaneous creation of particles do not occur.

On the other hand, if the Killing field is allowed to be space-like in certain regions, the above formalism is inapplicable[20] and the vacuum state need not be then stable. Although such situations are of great physical interest, a satisfactory formulation of quantum field theory on such space–times appears to be quite difficult. If the Killing field is allowed to be space-like, the energy associated with the classical fields would no longer be positive–definite, and hence, in general, well-behaved data sets for the Klein–Gordon fields would not necessarily evolve to non-singular solutions.[21] Under such pathologies in the Cauchy development, the very construction of the $*$-algebra $\mathscr{A}$ would break down. Hence, it is perhaps fair to say that, at the present stage, it is not clear how one would even *begin* to construct a quantum field theory on such space–times.[22]

5. General curved space–times

We now discuss the issue of obtaining the appropriate complex structure J in the case when the underlying space–time is not necessarily stationary.

Recall first that one expects from general physical considerations a non-stationary gravitational field to give rise to processes such as creation and annihilation of physical particles. On the other hand, it is clear from our discussion of quantum dynamics in §4, that, if we have a fixed, time-independent complex structure J, the vacuum state of the resulting Fock-representation is stable, i.e. there is no particle production whatsoever in the resulting quantum theory. Thus, general physical considerations suggest that, in the present case, we should try to obtain a complex structure which is, in a suitable sense, 'time-dependent'. A mathematical investigation of the possible complex structures supports this suggestion: there is apparently *no* natural time-independent complex structure on the vector space V of real solutions of the Klein–Gordon equation in a general non-stationary space–time.

Let us then try to select a 'time-dependent' complex structure. As one might expect from the fact that the underlying space–time is now arbitrary, the issue of this selection is quite complicated. It is therefore convenient to separate difficulties which are of purely mathematical origin from those associated with the physical interpretation. Hence, we shall proceed in the following steps: first, we shall concentrate only on the mathematical aspects of this problem, develop sufficient tools, prove certain existence and uniqueness theorems,[23] and then, at the end, face the issue of the physical interpretation of the resulting framework. The interpretation to which one is led is quite unconventional in certain respects and there still remain several issues to be resolved. We emphasize again that the discussion of this section should be regarded more as a first approach to the problem of describing quantum fields on general curved space–times, than as a completed theory in itself.

Introduce, on the given space–time, a time-like hypersurface orthogonal unit vector field ξ^a.[24] This vector field provides us with a slicing of space–time into space and time. It turns out that it also determines, canonically a time-dependent complex structure $J(t)$. More precisely, for each choice of a space-like hypersurface t_0, orthogonal to ξ^a, we can obtain a canonical complex structure $J(t_0)$, (compatible with the symplectic structure Ω), and hence, a canonical Fock representation of the $*$-algebra $\mathscr{A}$.

The essential idea is again to use an 'energy requirement' based on the intuitive notion of the wave-particle duality. Recall that, for this purpose we need two quantities: the energy E of the classical Klein–Gordon field and the Hamiltonian operator H on the one particle quantum states. In the previous two sections, we used the Killing field to define these two quantities. The idea now is to use the given vector field ξ^a. Consider, first, the energy: given a solution Φ of the Klein–Gordon equation, and a 3-surface t_0 orthogonal to ξ^a, we wish to define a number, E, the energy (relative to ξ^a) of the scalar field Φ at the instant of time t_0. We set

$$E(\xi^a, t_0) = \int_{t_0} T_{ab} N\xi^a \,\mathrm{d}S^b,$$

where T_{ab} is, as before, the stress energy constructed out of Φ, and where the scalar field N is defined by $N\xi^a\nabla_a t = 1$, with t the time-coordinate defined by ξ^a.[25] (Thus $N\xi^a \equiv \partial/\partial t$. Note that, if ξ^a happens to be a multiple of a Killing field, this E coincides with the usual energy which is then a conserved quantity.) The definition of the one particle Hamiltonian H, on the other hand, is not as straightforward. We cannot simply set $H\Phi^1 = -J(\mathscr{L}_{N\xi}\Phi)^1$ because, if $N\xi^a$ is not a Killing field, $\mathscr{L}_{N\xi}\Phi$ will not be in V for every Φ in V. Hence, to define H we must go back to the classical Hamiltonian and examine its action on the classical phase-space. This examination yields the following definition[26] of H: $H(\xi^a, t_0)\,\Phi^1 = -J(t_0)\,(\dot{\Phi}^{(t_0)})^1$, where, if Φ has the initial data $\Phi|t_0 = \varphi$, $\xi^a\nabla_a\Phi|_{t_0} = \pi$, then $\dot{\Phi}^{(t_0)}$ is the solution with initial data $\dot{\Phi}^{(t_0)}|_{t_0} = N\pi$, and $\xi^a\nabla_a\dot{\Phi}^{(t_0)}|_{t_0} = Nh^{ab}D_aD_b\varphi + h^{ab}D_aND_b\varphi - \mu^2N\varphi$. Here, h_{ab} is the induced metric on t_0, and D_a is the derivative operator on (t_0, h^{ab}). The dependence on t_0 of the 'dot' operation arises from the fact that the classical Hamiltonian is itself time-dependent in the present case. Thus, both E and H now depend on the choice of the vector field ξ^a as well as on that of the space-like 3-surface t_0. As before H depends, in addition, on the choice of the complex structure J. We now require that $J(t)$ be so chosen that $\langle\Phi^1, H(\xi^a, t)\,\Phi^1\rangle_t = E(\xi^a, t)$ for all ϕ in V; i.e., that the energy (relative to ξ^a) of each one-particle state be equal to that of the corresponding classical field at each instant of time t. As in the static case, using the expression of the inner-product $\langle,\rangle_t$ in terms of the symplectic structure Ω and the complex structure $J(t)$, this condition can be reduced to the following one: $\Omega(\Phi, J(t)\,\dot{\Phi}^{(t)}) = 0$ for all Φ in V.[16] It turns out that *this last condition selects for us, again, a unique complex structure* $J(t)$. The proof of this assertion is given in the appendix.

To summarize, given a time-like hypersurface orthogonal unit vector field[24] ξ^a, we obtain a canonical (time-dependent) structure $J(t)$, and hence, a one parameter family of Fock representations of the $*$-algebra $\mathscr{A}$.

How is dynamics to be described in the present case? As before, we require that the field operators should mimic the time-evolution of their classical counterparts, i.e. we set $(F(\Phi))^{\cdot}|_t = F(\dot{\Phi}^{(t)})$. Next, consider creation and annihilation operators. Since we have a Fock representation of $\mathscr{A}$ for each choice of the 3-surface t_0, we also have a decomposition of field operators into creation and annihilation parts for each such choice:

$$F(\Phi) = C(\Phi^1, t_0) + A(\Phi^1, t_0).$$

Hence,

$$C(\Phi^1, t) = (1/2i)\,[\mathrm{i}F(\Phi) + F(J(t)\,\Phi)] \quad \text{and} \quad A(\Phi^1, t) = (1/2\mathrm{i})\,[\mathrm{i}F(\Phi) - F(J(t)\,\Phi)].$$

Therefore, we have

$$[C(\Phi^1,t)]^{\cdot} = (1/2\mathrm{i})\left[\mathrm{i}F(\dot{\Phi}^{(t)}) + F\left(J(t)\,\dot{\Phi}^{(t)} + \left(\frac{\partial}{\partial t}J(t)\right)\Phi\right)\right]$$

and

$$[A(\Phi^1,t)] = (1/2\mathrm{i})\left[\mathrm{i}F(\dot{\Phi}^{(t)}) - F\left(J(t)\,\dot{\Phi}^{(t)} + \left(\frac{\partial}{\partial t}J(t)\right)\Phi\right)\right].$$

However, since in the present case, J is time-dependent, i.e. since in general

$$J(t_1)\,\Phi \neq J(t_2)\,(\Phi), \quad \text{or,} \quad \left(\frac{\partial}{\partial t}J(t)\right)\Phi \neq 0,$$

creation and annihilation operators 'get mixed' into each other as time evolves. Thus for example we have

$$[C(\Phi^1,t)]^{\cdot} = C(\dot{\Phi}^{(t)},t) + (1/2\mathrm{i})\,C\left(\left(\frac{\partial}{\partial t}J(t)\,\Phi\right)^1, t\right) + (1/2\mathrm{i})\,A\left(\left(\frac{\partial}{\partial t}J(t)\,\Phi\right)^1, t\right),$$

i.e. a creation operator picks up an annihilation part under the 'dot' operation. Hence, it follows that vacuum state (in any Fock space) is unstable: vacuum expectation values of (second and higher) time-derivatives of number operators fail to vanish. Thus, for example, if we evolve the number operator

$$N(\Phi,t_0) = C(\Phi,t_0)\,A(\Phi,t_0)$$

defined on the t_0 Fock-space, to time t_1, in general we would obtain, not the operator $N(\Phi,t_1)$, but rather, an operator whose expectation value in the vacuum state (of the t_1 Fock space) fails to vanish.

Note, however, that the dynamical rule, $(F(\Phi))^{\cdot}\,|_{t_0} = F(\dot{\Phi}^{(t_0)})$, is itself perfectly well-defined: it induces a one-parameter family of automorphisms[27] on the $*$-algebra $\mathscr{A}$. Since each Fock space gives us a well-defined $*$-representation of $\mathscr{A}$, and since the dynamical rule respects the structure of $\mathscr{A}$, all numbers which can be directly computed from this framework are necessarily divergence-free. More specifically, the expectation values of the rate of change of any operator which can be expressed as a finite sum of finite products of field operators (or equivalently of creation and annihilation operators) are guaranteed to be finite, Of course, there exist certain operators – notably the total-number operator[28] – which cannot be so expressed. The expectation values of the rate of change of such operators cannot be directly computed and must be examined separately.[29]

This completes our mathematical discussion. We now attempt to interpret this formalism physically. It seems natural to regard the elements of the Fock space associated with the 3-surface t as representing the 'quantum states of particles *at that instant of time*'. Failure of the complex structure to be time-independent then implies a spontaneous creation of particles. The 'rate of change of J with respect to time' is then a measure of the 'rate of production of particles': the inner product

$$\left\langle \left[\left(\left(\frac{\partial}{\partial t}J(t)\right)\Phi\right)^1, t\right],\ \left[\left(\left(\frac{\partial}{\partial t}J(t)\right)\Phi\right)^1, t\right]\right\rangle_t$$

governs the rate at which the particles are produced in the state $[\Phi, t]$ at the instant of time t.

However, if one accepts this interpretation, one is led to an unconventional picture of physical particles: since the choice of the complex structure depends quite crucially on that of the (time-like) vector field ξ^a, one obtains a definition of particle-states (or, equivalently of the vacuum) for each choice of this ξ^a. That is to say, the above mathematical framework seems to imply that the notion of a particle be a 'relative' notion rather than a fixed, absolute one.

Let us, for the moment, accept this viewpoint and explore some of its implications. First, since it is a vector field ξ^a that determines the one parameter family of Fock representations, and since the integral curves of such a ξ^a can be thought of as representing the world-lines of a field of observers, perhaps the most natural interpretation is that, in this framework, the notion of a particle would depend on the choice of observers. Recall, however, that, in the previous section, we obtained a fixed, canonical representation of $\mathscr{A}$, and hence, a fixed, canonical notion of particles in any given stationary space–time. How is that result to be interpreted in the light of the present discussion? Recall that this canonical representation was obtained by using the Killing field t^a in place of $N\xi^a$. Hence, one could interpret the elements of that Fock space as being the quantum states of particles *as seen by the observers following the Killing trajectories*. From this viewpoint, the stationary Fock space is *canonical* or *natural* only in so far as the motions along Killing trajectories are *canonical* or *natural*. Thus, one can, even in the stationary space–time, make a different choice for the vector field ξ^a and obtain a completely different notion of particles. In particular, already in Minkowski space–time, one could choose for ξ^a a vector field which fails to be a Killing field and obtain a representation in which the vacuum is unstable: the present formalism seems to predict that a detector moving along such a vector field would observe, at least locally, particles being spontaneously created and annihilated.

This 'observer dependent' framework, although quite unconventional, is perhaps not a hopeless candidate for describing quantum fields in curved space–times. For example, within this framework, one would still be able to formulate the various conservation laws which will be respected for any choice of the field of observers. It may also turn out that each set of observers will find the world around themselves not only consistent with the theoretical predictions but also quite 'normal': for example, non-stationary observers in stationary space–times will 'experience' a time dependence in the space–time metric, and hence, will not be surprised to 'see' also a spontaneous creation of particles. Unconventional features (such as the 'relativity' in the notion of particles) arise only when we compare the predictions for two *different* sets of observers. Note, furthermore, that these 'unconventional' features do not contradict, e.g. the usual 'Poincaré invariant physics': it is in fact a *prediction* of this frame-work, that, in Minkowski space, all inertial observers would 'see the same physics'. Unconventional features refer only to situations in which we allow non-inertial observers in Minkowski space, or, more generally, non-stationary observers in stationary space–times. Furthermore, these features are in (at least qualitative) agreement with certain recent suggestions (Unruh 1974*b*) based on general physical considerations: using simple models of detectors it has been argued that, in Minkowski space, accelerating observers would detect particles being spontaneously created and annihilated even when the inertial observers record only a (stable) vacuum. Finally, this 'observer dependent' description has a certain asthetic appeal: it sets up some sort of a tie between what the observers do and what they detect. One would hope that some such tie would be retained in the 'ultimate' quantum theory, should there exist such a theory.

6. Discussion

In §1 we began our discussion by asking why, in the standard formulation of the quantum field theory, a 'global' technique such as the decomposition of fields into positive and negative frequency parts is required for a successful description of such 'local' entities as particles. The general mathematical analysis of §2 led us to the conclusion that this decomposition serves precisely one purpose: it selects a suitable complex structure J on the vector space of real solutions of the Klein–Gordon equation. We therefore focused our attention on this J and tried to understand why, among all possible complex structures, it is the one which is physically appropriate. The energy-requirement suggested an answer: one might say that this choice of J is the physically correct one because only for this choice is the energy of each one-particle quantum state equal to that of the corresponding classical field. Thus, the success of the technique of decomposing fields can be traced back to the interplay between classical waves and quantum (states of) particles; an interplay which dominates the mathematics of quantum field theory.

Since, in classical general relativity, we can introduce the notion of energy of a system relative to any given set of observers – not just those following Killing trajectories – we can obtain a (modified) energy-requirement for any given set of observers in any given space–time. Having adopted the viewpoint that energy-requirements are somehow fundamental in the choice of the complex structures, we are then led to a 'observer-dependent' complex structure and hence to a 'observer-dependent' notion of particles. Although this dependence on the choice of observers is unconventional, the resulting formalism is of interest because it makes definite predictions. Furthermore, these predictions refer, not to asymptotic measurements, but rather to the detailed measurements made by observers travelling through regions in which space–time curvature is not necessarily negligible. In addition, there are predictions about, for example, what non-inertial observers would see already in Minkowski space.

Of course there remain several issues which must be investigated in detail before one can actually accept that such observer-dependent predictions refer to the 'real world'.

We shall conclude this discussion by describing what is perhaps the most obvious of such issues. Consider a zero rest mass field in an asymptotically simple space–time. In this case, one can construct a Fock space of ingoing states on $\mathscr{I}^-$, a Fock space of outgoing states on $\mathscr{I}^+$, and obtain an unambiguous S-matrix description of the various quantum processes such as the net creation and annihilation of particles by the underlying gravitational field. Note that, this framework refers only to the intrinsic geometrical structure of the given space–time (and of $\mathscr{I}$) and does not require, in addition, a preferred vector field: the predictions of this framework refer only to asymptotic measurements. Hence, if the present 'observer-dependent' description is to be viable, its predictions about these *asymptotic measurements* should be independent of the choice of the field of observers (i.e. of the time-like vector field). Perhaps the most striking drawback of the present description is that, it is not clear, *a priori*, if it satisfies this viability criterion. So far, a detailed examination of the situation has yielded only the following result: if two vector fields coincide outside a 'finite time-interval', then, although the respective quantum descriptions (cf. § 5) will, in general, be quite different from each other within that interval, their S-matrix descriptions in terms of 'ingoing' and 'outgoing' states (with respect to that interval) will coincide.[30] This result itself can only be regarded as an indication that the viability criterion under discussion will probably be satisfied. A more detailed investigation is necessary to settle even this issue.

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Appendix

Consider a space–time (M, g_{ab}). Choose a space-like Cauchy surface t_0 in this space–time. Let h_{ab} denote the induced metric on this t_0; and let D_a denote the derivative operator and $\mathrm{d}V$ the volume element on (t_0, h_{ab}). Next, consider the vector space $\hat{V}$ of pairs, (φ, π) of functions which are real, smooth, and of compact support on t_0. Define on this $\hat{V}$ a 2-form as follows: $\hat{\Omega}((\varphi,\pi),(\tilde{\varphi},\tilde{\pi})) = \int_{t_0} (\pi\tilde{\varphi} - \varphi\tilde{\pi})\,\mathrm{d}V$, for any two elements (φ,π) and $(\tilde{\varphi},\tilde{\pi})$ of $\hat{V}$.

The vector space $\hat{V}$ is of course naturally isomorphic to the space V of all well behaved, real solutions of the Klein–Gordon equation. This natural isomorphism maps $\hat{\Omega}$ on $\hat{V}$ to the 2-form Ω on V defined in § 1. We now give a proof of the assertion about the uniqueness of the complex structure on V (made in § 5) in terms of $\hat{V}$ and $\hat{\Omega}$.

Theorem. *There exists a unique complex structure $\hat{J}$ on $\hat{V}$ such that*

(i) $\hat{\gamma} = \hat{\Omega} \ulcorner \hat{J}$ *is a positive definite metric on* $\hat{V}$; *and*

(ii) $\hat{\gamma}((\varphi,\pi),(\dot{\varphi},\dot{\pi})) = 0$ *for every pair* (φ,π) *in* $\hat{V}$, *where* $(\dot{\varphi},\dot{\pi})$ *is related to* (φ,π) *by* $\dot{\varphi} = N\varpi$, $\dot{\pi} = ND^aD_a\varphi + D^aND_a\varphi - \mu^2N\varphi$; *$N$ being any fixed positive function on* t_0, *and* μ, *any fixed real number.*

We now give the main steps in the proof.

Let τ denote the space of all real-valued, smooth functions of compact support on t_0. Since $\hat{J}$ is a linear-operator on $\hat{V}$ there exist linear operators A, B, C, and D on τ such that $\hat{J}(\varphi, \pi) = (A\varphi + B\pi, C\pi + D\varphi)$. We shall now show that conditions (i) and (ii), given above, determine these operators completely.

Since $\hat{J}$ is a complex structure, we have $\hat{J}^2 = -I$. Hence, it follows that

$$\left.\begin{aligned} A^2+BD = -I \quad &\text{and} \quad AB+BC = 0,\\ C^2+DB = -I \quad &\text{and} \quad DA+CD = 0.\end{aligned}\right\} \tag{A 1}$$

Next, we incorporate conditions (i) and (ii). For this purpose, we must first express $\hat{\gamma}$ in terms of A, B, C, and D. Using, in the definition of $\hat{\gamma}$, the expression for $\hat{\Omega}$, we obtain

$$\hat{\gamma}((\varphi,\pi)(\tilde{\varphi},\tilde{\pi})) = \int_{t_0} (\pi B\tilde{\pi} - \varphi D\tilde{\varphi} + \pi A\tilde{\varphi} - \varphi C\tilde{\pi})\,\mathrm{d}V \tag{A 2}$$

for all pairs (φ,π) and $(\tilde{\varphi},\tilde{\pi})$. Since $\hat{\gamma}$ is symmetric, it follows that

$$\left.\begin{aligned} &\int_{t_0} \tilde{f}Bf\,\mathrm{d}V = \int_{t_0} fB\tilde{f}\,\mathrm{d}V, \quad \int_{t_0} \tilde{f}Df\,\mathrm{d}V = \int_{t_0} fD\tilde{f}\,\mathrm{d}V\\ \text{and} \quad &\int_{t_0} \tilde{f}Af\,\mathrm{d}V = -\int_{t_0} fC\tilde{f}\,\mathrm{d}V\end{aligned}\right\} \tag{A 3}$$

for all elements f and $\tilde{f}$ of τ. Next, it follows from positive definiteness of $\hat{\gamma}$ that,

$$\int_{t_0} fBf\,\mathrm{d}V > 0 \quad \text{and} \quad \int_{t_0} fDf\,\mathrm{d}V < 0 \tag{A 4}$$

for all f in τ. Finally, since $\hat{\gamma}((\varphi,\pi),(\dot{\varphi},\dot{\pi})) = 0$, for all φ and π in τ, we have

$$\int_{t_0} Nf(Cf)\,\mathrm{d}V = 0 \quad \text{and} \quad \int_{t_0} (Nf)(D\tilde{f})\,\mathrm{d}V = -\int_{t_0} (N^{-1}\Theta\tilde{f})\,Bf\,\mathrm{d}V, \tag{A 5}$$

where we have set $N^2D^aD_a + ND^aND_a - \mu^2N^2 = -\Theta$.

We have now imposed all available conditions on A, B, C, and D. First, we shall show that these conditions imply $A = 0$ and $C = 0$. Setting $f = g+h$ in the first of the equations (A 5) and using the last of the equations (A 3) we obtain

$$C = N^{-1}AN \tag{A 6}$$

as operators on τ. Hence, it suffices to show that $A = 0$. For this purpose we introduce a Hilbert space K: K is the Cauchy completion of the inner-product space $(\tau, (,))$, where $(f,g) = \int_{t_0} fgN^{-1}\,\mathrm{d}V$. Clearly, operators A, B, C and D are all densely defined on this K. Furthermore, it follows from (A 3) and (A 4) that operators $N.B$ and $-N.D$ are both symmetric and positive definite. Let us assume $A \neq 0$. Then, for every f and $\tilde{f}$ in τ which are not annihilated by A, we have

$$(\tilde{f}, A^2f) = \int_{t_0} \tilde{f}(A\,.\,Af)\,N^{-1}\,\mathrm{d}V = -\int_{t_0} C(N^{-1}\tilde{f})\,(Af)\,\mathrm{d}V = -\int_{t_0} (A\tilde{f})\,(Af)\,N^{-1}\,\mathrm{d}V.$$

(We have used the last of the equations (A 3) and (A 6).) Thus, on the subspace S_A of K which consists of all f's with $Af \neq 0$, A is a negative definite, symmetric operator. Similarly, by using (A 6), the last of the equations (A 3) and (A 1), it is easy to show that $(-N.D).A^2$ is a positive definite symmetric operator. Thus, on S_A, we have symmetric operators $(-N.D)$, A^2 and $(-N.D).A^2$, with $(-N.D)$ positive definite, A^2 negative definite and $(-N.D).A^2$ positive definite. This implies that S_A must be empty. Thus, $A = 0$, and consequently, $C = N^{-1}AN = 0$.

Next, we consider B and D. From the second of the equations (A 5) it follows that $D = -N^{-1}BN^{-1}\Theta$, while from the first of (A 1) we have $D.B = B.D = -I$. Using the fact that Θ is an essentially self-adjoint positive-definite operator[17] on K, one obtains $D = -N^{-1}\Theta^{\frac{1}{2}}$, where $\Theta^{\frac{1}{2}}$ is itself a positive-definite, self-adjoint operator, the positive square-root of Θ on K. Finally, since $B.D = D.B = -I$, we have $B = \Theta^{-\frac{1}{2}}N$.

Thus $\hat{J}(\varphi, \pi) = (\Theta^{-\frac{1}{2}}N\pi, -N^{-1}\Theta^{\frac{1}{2}}\varphi)$ is the only complex structure on $\hat{V}$ which satisfies the conditions of the theorem.

Finally, we remark that, if (M, g_{ab}) happens to be a static space-time, t_0 a static surface, N the norm of the static Killing field, then the complex structure J induced on V by the above $\hat{J}$ on $\hat{V}$ (under the natural isomorphism between $\hat{V}$ and V) is the same as the one defined in terms of the standard positive and negative frequency decompositions.

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Notes

[1] We shall assume throughout that the Cauchy problem for the classical Klein–Gordon equation is well posed on the given space–time. More precisely, we shall assume that the space–time admits a space-like Cauchy surface and that every smooth initial data set with compact support on such a Cauchy surface evolves to a (everywhere) smooth solution.

[2] A complex structure J on a real vector space V is a linear operator on V satisfying $J^2 = -I$.

[3] That is to say, space–times which admit a Killing vector which is *everywhere* time-like and hypersurface orthogonal.

[4] That is to say, space–times which admit a Killing vector which is *everywhere* time-like.

[5] In this section the space–time is allowed to be arbitrary. The only restriction we shall make is that the Cauchy problem for the classical Klein–Gordon equation be well posed.

[6] That is to say, solutions which are smooth and induce, on any space-like Cauchy surface, initial data sets of compact support. The assumption about compact support is just for mathematical convenience and may be replaced by the requirement that the data sets go to zero sufficiently fast (as we approach spatial infinity) for various integrals which arise in this discussion to converge.

[7] The phrase ‘impose the condition $A = 0$ on the given $*$-algebra’ will mean ‘consider the $*$-ideal generated by A and take the quotient of the given $*$-algebra by this ideal’.

[8] Since the integrand is divergence-free and since it vanishes at spatial infinity, the integral is independent of the particular choice of t_0 made in its evaluation.

[9] A $*$-representation of a $*$-algebra $\mathscr{A}$ is a map Λ from $\mathscr{A}$ into the $*$-algebra of operators defined on a Hilbert space $\mathscr{F}$ satisfying $\Lambda(A+kB) = \Lambda(A)+k\Lambda(B)$: $\Lambda(AB) = \Lambda(A)\,\Lambda(B)$; and $\Lambda(A^*)$ is the adjoint of $\Lambda(A)$ on $\mathscr{F}$; for all A and B in $\mathscr{A}$ and for all complex numbers k.

[10] Actually, this requirement follows from the other two if we demand that the $*$-representation be faithful. Although the faithfulness requirement is perhaps better motivated from a physical viewpoint, we have chosen to impose in its stead the condition on the ‘size’ of $\mathscr{H}$ because it is this condition which will enter directly in the subsequent discussion.

[11] Our notation is the following: Φ^1 is that element of the one-particle Hilbert space $\mathscr{H}$ which is defined by the classical solution Φ of the Klein–Gordon equation.

[12] That is to say (V, J, Ω, γ) is a Kähler space.

[13] If J ‘changes in time’ creation and annihilation operators ‘get mixed’ as time evolves. This ‘mixing’ is often described in terms of a non-trivial Bogoliubov transformation (see, for example, De Witt 1975).

[14] The equation $H\Phi^1 = -J(\mathscr{L}_t\Phi)^1$ may also be regarded as Schrödinger’s evolution equation on the Hilbert space $\mathscr{H}$.

[15] An alternative proof is given in § 3 in the slightly more general context of stationary space–times.

[16] Thus, alternatively, we could have proceeded as follows: instead of the energy requirement, we could have required only that $\langle\Phi^1, H\Phi^1\rangle$ be real, shown that there exists a unique complex structure J (compatible with the symplectic structure Ω) for which this requirement is satisfied, and then shown, at the end, that the energy requirement is automatically satisfied by this J.

[17] From the definition of Θ it follows, directly, that it is a (densely defined) symmetric, positive-definite operator bounded below. Hence, it admits a maximal self-adjoint extension, namely the Friedrichs extension (see, for example, Gelfand & Shilov 1967). Operators $\Theta^{\frac{1}{2}}$ and $\Theta^{-\frac{1}{2}}$ are defined by using the spectral decomposition of Θ (see for example, Akhiezer & Glazman 1963).

[18] Consider the case when (M, g_{ab}, t^a) is a static space–time. In this case, we can decompose any given solution into normal modes. Then, $\mathscr{L}_t$ multiplies each normal mode of frequency ω by $\mathrm{i}\omega$ and $\Theta^{-\frac{1}{2}}$ divides each such mode by $|\omega|$. Thus, $\Theta^{-\frac{1}{2}}\mathscr{L}_t$ multiplies the positive frequency part of every real solution Φ by i and the negative frequency part by $(-\mathrm{i})$; i.e. $\Theta^{\frac{1}{2}}\mathscr{L}_t$ is indeed the standard complex structure.

[19] Since Θ is a positive definite self-adjoint operator which commutes with $\mathscr{L}_t$, it follows that $\Theta^{-\frac{1}{2}}$ also commutes with $\mathscr{L}_t$. Hence, $J^2 = \Theta^{-\frac{1}{2}}\mathscr{L}_t\Theta^{-\frac{1}{2}}\mathscr{L}_t = \Theta^{-1}\mathscr{L}_t\mathscr{L}_t = -I$. Thus, J is indeed a complex structure. Next, since $(\Phi, \tilde{\Phi}) = \Omega(\Phi, \mathscr{L}_t\tilde{\Phi})$ for all Φ and $\tilde{\Phi}$ in V, we have,

$$(\Phi, \Theta^{-\frac{1}{2}}\tilde{\Phi}) = \Omega(\Phi, \mathscr{L}_t\Theta^{-\frac{1}{2}}\tilde{\Phi}) = \Omega \ulcorner J(\Phi, \tilde{\Phi}).$$

Thus, $\Omega \ulcorner J$ is a positive definite metric on V.

[20] In this case, the norm, (Φ, Φ) induced by the inner product will no longer be positive definite and hence our construction of J will not go through.

[21] The following argument due to Friedman (1974) indicates that such instabilities will occur quite generically in space–times which have ergospheres but no horizon. Consider an initial data set with support inside the ergosphere and with negative total energy. As time evolves, the field will escape out of the ergosphere. (For example, if it is a zero rest mass field, it will escape to $\mathscr{I}$). The part of the field which escapes carries with it a positive energy. Since the total energy is conserved, the contribution to the energy from within the ergosphere would become more and more negative, i.e. in the ergosphere, the field would grow unboundedly in time.

[22] Note that we are considering here a generic situation. In particular space–times, e.g. in the Kerr solutions, it may turn out that the Cauchy development (of the test-fields) is regular and one might then be able to invent ways to describe quantum fields on such space–times.

[23] Several of these mathematical results were briefly reported elsewhere. (Ashtekar & Magnon 1975).

[24] Alternatively, we can introduce a scalar field t with $\nabla_a t$ everywhere time-like, and set $\xi^a = -N\nabla^a t$, where $N = (\nabla^a t\, \nabla_a t)^{-\frac{1}{2}}$. Then, $N\xi^a \nabla_a t = 1$ and hence we can identify $N\xi^a$ with $\partial/\partial t$. Thus, effectively, here we have set the lapse field equal to N and the shift field equal to zero. We see no essential difficulty in the incorporation of the non-zero shifts, i.e. in allowing ξ^a to be non-hypersurface orthogonal. What we really need here is only a foliation of space-time.

[25] Note that, as in the static case, the vector field which appears in the integrand is $\partial/\partial t$ $(\equiv N\xi^a)$ rather than the unit field $N^{-1}\,\partial/\partial t\ (\equiv \xi^a)$. $E(\xi^a, t)$ is the natural candidate for the classical energy because, when regarded as a function on phase-space, it coincides with the classical Hamiltonian, i.e., with the generator of evolution with respect to the time-parameter defined by ξ^a.

[26] In the 2-component wave-function formalism (see, for example, Unruh 1974*a*) this definition of $H(\xi^a, t_0)$ emerges as the most natural one.

[27] More generally, any transformation on $\mathscr{A}$ which is induced by a linear canonical transformation on the classical phase-space, is an automorphism (Segal 1967).

[28] Even if it should turn out that the expectation values of the rate of change of the total-number operator are not necessarily finite, it would not be very disturbing: such a pathology would probably have its origin in our ‘external field approximation’ (i.e. in our insistence of keeping the background geometry fixed), and would therefore disappear when this approximation is removed.

[29] These operators appear as an infinite convergent series of (finite) products of field operators. (Thus, they are elements of an appropriate completion of $\mathscr{A}$ and not of $\mathscr{A}$ itself.) The question is if the evolution rule respects the topology on $\mathscr{A}$ in which the relevant sum converges.

[30] Thus, in particular, if in Minkowski space, we choose a field of observers which is non-inertial only in some compact region, then the prediction of the framework discussed in §5 is that, although these observers would see, locally, particles being spontaneously created and annihilated in that region, they would see no net particle production.